Transformational principles latent in the theory of

CLIFFORD ALGEBRAS

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Introduction. My purpose in this informal material will be to refresh/consolidate my own thought concerning a subject that has engaged my attention from time to time over the years, but not recently. I do so because aspects of the subject have become the focus of a thesis effort by Nakul Shankar, whose awkward situation is that he is going to have to change horses in midstream: I will direct the first phase of his project, but will be obliged to pass the baton to Tom Wieting & Darrell Schroeter at mid-year. A large part of my effort here, therefore, will be directed to the the establishment of some degree of notational and conceptual commonality, and to construction of a clear statement of my own motivating interests in this area.

1. Complex algebra, revisited. Familiarly, $x^2 + y^2$ does not factor on the reals, but the related object $(x^2 + y^2)\mathsf{I}$ does factor

$$(x^2 + y^2)\mathsf{I} = (x\mathsf{I} + y\boldsymbol{i}) \cdot (x\mathsf{I} - y\boldsymbol{i}) \tag{1}$$

provided I and $\boldsymbol{i}$ are objects with the stipulated properties

$$\left.\begin{aligned} \mathsf{I}\cdot\mathsf{I} &= \mathsf{I} \\ \mathsf{I}\cdot\boldsymbol{i} &= \boldsymbol{i} \\ \boldsymbol{i}\cdot\mathsf{I} &= \boldsymbol{i} \\ \boldsymbol{i}\cdot\boldsymbol{i} &= -\mathsf{I} \end{aligned}\right\} \tag{2}$$

Equations (2) are collectively equivalent to the statement that if $\boldsymbol{z}_1 = x_1\mathsf{I} + y_1\boldsymbol{i}$ and $\boldsymbol{z}_2 = x_2\mathsf{I} + y_2\boldsymbol{i}$ then

$$\boldsymbol{z}_1 \cdot \boldsymbol{z}_2 = (x_1x_2 - y_1y_2)\mathsf{I} + (x_1y_2 + y_1x_2)\boldsymbol{i} \tag{3}$$

Evidently

$$\boldsymbol{z}_1 \cdot \boldsymbol{z}_2 = \boldsymbol{z}_2 \cdot \boldsymbol{z}_1 \quad : \quad \left\{\begin{array}{l}\text{The algebra is } \textbf{commutative}\text{, and is found} \\ \text{by calculation to be also } \textbf{associative}.\end{array}\right. \tag{4}$$

It is evident also that $x_1y_2 + y_1x_2 = 0$ iff $x_1/y_1 = -x_2/y_2$: we are motivated therefore to introduce the operation

$$\boldsymbol{z} = x\mathbf{I} + y\boldsymbol{i} \xrightarrow[\text{conjugation}]{} \overline{\boldsymbol{z}} = x\mathbf{I} - y\boldsymbol{i} \tag{5}$$

Then

$$\boldsymbol{z}\cdot\overline{\boldsymbol{z}} = (x^2 + y^2)\mathbf{I} \tag{6}$$

and we find that

$$\boldsymbol{z}^{-1} = \frac{\overline{\boldsymbol{z}}}{x^2 + y^2} \quad \text{exists unless } x^2 + y^2 = 0;\ \textit{i.e.}, \text{ unless } \boldsymbol{z} = \mathbf{0} \tag{7}$$

We agree to call

$$|\boldsymbol{z}| \equiv \sqrt{x^2 + y^2} \geqslant 0 \tag{8}$$

the "modulus" of $\boldsymbol{z}$. By calculation we discover that

$$|\boldsymbol{z}_1\cdot\boldsymbol{z}_2| = |\boldsymbol{z}_1|\cdot|\boldsymbol{z}_2| \tag{9}$$

We can mechanize the condition that $\boldsymbol{z}$ be "unimodular" ($|\boldsymbol{z}| = 1$) by writing

$$\boldsymbol{z} = \cos\theta\cdot\mathbf{I} + \sin\theta\cdot\boldsymbol{i} = e^{\boldsymbol{i}\theta} \tag{10}$$

Transformations of the form

$$\boldsymbol{z} \longmapsto \boldsymbol{Z} \equiv e^{\boldsymbol{i}\theta}\cdot\boldsymbol{z} = (x\cos\theta - y\sin\theta)\mathbf{I} + (x\sin\theta + y\cos\theta)\boldsymbol{i} \tag{11}$$

are manifestly modulus-preserving. Notated

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{12}$$

they have clearly the structure characteristic of *rotations*. Writing

$$\mathbb{R}(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \equiv \cos\theta\cdot\mathbb{I} + \sin\theta\cdot\mathbb{J} \tag{13}$$

we arrive at a 2×2 *matrix representation* of the algebra now in hand. The matrices

$$\mathbb{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{J} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

are readily seen to satisfy (compare (2))

$$\left.\begin{aligned} \mathbb{I}\cdot\mathbb{I} &= \ \ \mathbb{I} \\ \mathbb{I}\cdot\mathbb{J} &= \ \ \mathbb{J} \\ \mathbb{J}\cdot\mathbb{I} &= \ \ \mathbb{J} \\ \mathbb{J}\cdot\mathbb{J} &= -\mathbb{I} \end{aligned}\right\} \tag{15}$$

so we are led to the identification

$$\boldsymbol{z} = x\,\mathbf{l} + y\,\boldsymbol{i} \;\longleftrightarrow\; \mathbb{Z} = x\,\mathbb{I} + y\,\mathbb{J} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \tag{16}$$

In this representation

$$\text{conjugation} \;\longleftrightarrow\; \text{transposition} \tag{17}$$

and

$$|\boldsymbol{z}|^2 = \det\mathbb{Z} \tag{18}$$

Alternative matrix representations can be obtained by similarity transformation

$$\mathbb{Z} \;\longrightarrow\; \mathbb{Z}' \equiv \mathbb{S}^{-1}\mathbb{Z}\,\mathbb{S} \tag{19}$$

Such transformations preserve (15) and (18), and preserve also the spectral features of $\mathbb{Z}$, which are instructive, and to which I now turn: the

$$\begin{aligned} \text{characteristic polynomial} &= \lambda^2 - 2x\lambda + (x^2+y^2) \\ &= \lambda^2 - \operatorname{tr}\mathbb{Z}\cdot\lambda + \det\mathbb{Z} \\ &= \lambda^2 - \operatorname{tr}\mathbb{Z}\cdot\lambda + \tfrac{1}{2}\big\{\operatorname{tr}\mathbb{Z}^2 - (\operatorname{tr}\mathbb{Z})^2\big\} \end{aligned}$$

so by the Cayley-Hamilton theorem we have $\mathbb{Z}^2 - 2x\,\mathbb{Z} + (x^2+y^2)\mathbb{I} = \mathbb{O}$ whence

$$\mathbb{Z}^{-1} = \frac{2x\,\mathbb{I} - \mathbb{Z}}{x^2+y^2}$$

But $2x\,\mathbb{I} - \mathbb{Z} = \mathbb{Z}^{\mathsf{T}}$, so we have

$$= \frac{\mathbb{Z}^{\mathsf{T}}}{\det\mathbb{Z}}$$

which is the matrix representation of (7). The eigenvalues of $\mathbb{Z}$ are $x \pm iy$ and the associated eigenvalues are $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$, which is to say: we have

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} \pm i \\ 1 \end{pmatrix} = (x \pm iy)\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$$

I hope my reader will forgive me for belaboring the familiar: my effort has been to establish a pattern, the first rough outline of a template to which we can adhere when we turn to less familiar subject matter.

2. Clifford algebra of order 2. This subject arises when we ask not—as at (1)—to factor but to *extract the formal square root* of $x^2 + y^2$. Or, as we find it now more convenient to notate the assignment, to extract the square root of

$$x^1x^1 + x^2x^2 \equiv \delta_{ij}x^ix^j \quad \text{where} \quad \|\delta_{ij}\| \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To that end we posit the existence of objects I, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ such that

$$(\delta_{ij}x^ix^j)\mathsf{I} = (x^i\boldsymbol{e}_i)^2 \quad : \quad \text{all } x^i \tag{20}$$

Immediately

$$\begin{aligned} \boldsymbol{e}_i\,\boldsymbol{e}_j + \boldsymbol{e}_j\,\boldsymbol{e}_i &= (\delta_{ij} + \delta_{ji})\mathsf{I} \\ &= 2\delta_{ij}\mathsf{I} \quad \text{by} \quad \delta_{ij} = \delta_{ji} \end{aligned} \tag{21}$$

which when spelled out in specific detail read

$$\boldsymbol{e}_1\,\boldsymbol{e}_1 = \boldsymbol{e}_2\,\boldsymbol{e}_2 = \mathsf{I} \tag{22.1}$$

$$\boldsymbol{e}_1\,\boldsymbol{e}_2 + \boldsymbol{e}_2\,\boldsymbol{e}_1 = \mathbf{0} \tag{22.2}$$

It follows that products of the general form $\boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\boldsymbol{e}_{i_3}\cdots\boldsymbol{e}_{i_n}$, which we suppose to have been assembled from p $\boldsymbol{e}_1$'s and $q = n-p$ $\boldsymbol{e}_2$'s, can (by $\boldsymbol{e}_2\,\boldsymbol{e}_1 = -\,\boldsymbol{e}_1\boldsymbol{e}_2$) always be brought to "dictionary order"

$$\pm\,\underbrace{\boldsymbol{e}_1\boldsymbol{e}_1\cdots\boldsymbol{e}_1}_{p\ \text{factors}}\underbrace{\boldsymbol{e}_2\boldsymbol{e}_2\cdots\cdots\boldsymbol{e}_2}_{q\ \text{factors}}$$

where $(\pm) = (-)^{\text{number of transpositions required to achieve dictionary order}}$. Drawing now upon (22.1) we find that the expression presented just above can be written

$$= \pm\begin{cases} \mathsf{I} & \text{if } p \text{ even, } q \text{ even} \\ \boldsymbol{e}_1 & \text{if } p \text{ odd, } q \text{ even} \\ \boldsymbol{e}_2 & \text{if } p \text{ even, } q \text{ odd} \\ \boldsymbol{e}_1\boldsymbol{e}_2 & \text{if } p \text{ odd, } q \text{ odd} \end{cases}$$

and that this list exhausts the possibilities. We confront therefore an algebra with elements of the form

$$\boldsymbol{a} = a^0\mathsf{I} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^{12}\boldsymbol{e}_1\boldsymbol{e}_2 \tag{23}$$

If $\boldsymbol{b}$ is defined similarly then, by computation,

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b} = \quad & (a^0b^0 + a^1b^1 + a^2b^2 - a^{12}b^{12})\,\mathsf{I} \\ &+ (a^0b^1 + a^1b^0 - a^2b^{12} + a^{12}b^2)\,\boldsymbol{e}_1 \\ &+ (a^0b^2 + a^1b^{12} + a^2b^0 - a^{12}b^1)\,\boldsymbol{e}_2 \\ &+ (a^0b^{12} + a^1b^2 - a^2b^1 + a^{12}b^0)\,\boldsymbol{e}_1\boldsymbol{e}_2 \end{aligned} \tag{24}$$

from which it follows as a corollary that

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}-\boldsymbol{b}\cdot\boldsymbol{a} = \;& 2(-a^2b^{12}+a^{12}b^2)\,\boldsymbol{e}_1\\ &+2(+a^1b^{12}-a^{12}b^1)\,\boldsymbol{e}_2\\ &+2(+a^1b^2\;-a^2b^1\;)\,\boldsymbol{e}_1\boldsymbol{e}_2\end{aligned}\tag{25}$$

Equation (24) serves in effect as a "multiplication table," while (25) makes vivid the fact—evident already in (22)—that we have now in hand an algebra that is (as calculation would confirm) **associative** but **non-commutative**. If we let the "conjugate" of $\boldsymbol{a}$ be defined/denoted

$$\overline{\boldsymbol{a}} = a^0\mathbf{I} - a^1\boldsymbol{e}_1 - a^2\boldsymbol{e}_2 - a^{12}\boldsymbol{e}_1\boldsymbol{e}_2\tag{26}$$

then it follows from (24) that

$$\boldsymbol{a}\cdot\overline{\boldsymbol{a}} = (a^0a^0 - a^1a^1 - a^2a^2 + a^{12}a^{12})\,\mathbf{I}\tag{27}$$

and from (25) that

$$= \overline{\boldsymbol{a}}\cdot\boldsymbol{a}$$

Evidently a right/left inverse of $\boldsymbol{a}$ exists iff the "modulus" of $\boldsymbol{a}$

$$|\,\boldsymbol{a}\,| \equiv a^0a^0 - a^1a^1 - a^2a^2 + a^{12}a^{12}\tag{28}$$

does not vanish, and is given then by

$$\boldsymbol{a}^{-1} = \frac{\overline{\boldsymbol{a}}}{|\,\boldsymbol{a}\,|}\tag{29}$$

By *Mathematica*-assisted calculation we establish that

$$|\boldsymbol{a}\cdot\boldsymbol{b}| = |\boldsymbol{a}\,|\cdot|\boldsymbol{b}|\tag{30}$$

Transformations of the form

$$\boldsymbol{a} \;\longmapsto\; \boldsymbol{A} = \boldsymbol{u}^{-1}\boldsymbol{a}\,\boldsymbol{u}\tag{31}$$

are therefore modulus-preserving, and we can in such a context assume without loss of generality that $\boldsymbol{u}$ is unimodular: $|\boldsymbol{u}| = 1$. Equation (31) serves to establish a linear relationship between the coefficients of $\boldsymbol{A}$ and those of $\boldsymbol{a}$:

$$\begin{pmatrix}A^0\\A^1\\A^2\\A^3\end{pmatrix} = \mathbb{U}\begin{pmatrix}a^0\\a^1\\a^2\\a^3\end{pmatrix}\tag{32}$$

Notational remark: I have at this point found it convenient to write a^3 in place of a^{12}, $\boldsymbol{e}_3$ in place of $\boldsymbol{e}_1\boldsymbol{e}_2$, *etc.*

One could—quickly enough, with the assistance of *Mathematica*—work out explicit descriptions of the elements of $\mathbb{U}$ (they are assembled quadratically from the elements of $\boldsymbol{u}$), but it is simpler and more sharply informative to proceed on the assumption that $\boldsymbol{u}$ differs only infinitesimally from I:[1]

$$\boldsymbol{u} = \mathsf{I} + \boldsymbol{w} \quad : \quad \text{terms of } 2^{\text{nd}} \text{ order in } \boldsymbol{w} \text{ will be neglected}$$

Then $\boldsymbol{u}^{-1} = \mathsf{I} - \boldsymbol{w}$ in leading order, which on comparison with $\boldsymbol{u}^{-1} = \overline{\boldsymbol{u}}$ means that we can without loss of generality assume that $w^0 = 0$:

$$\boldsymbol{w} = w^1\boldsymbol{e}_1 + w^2\boldsymbol{e}_2 + w^3\boldsymbol{e}_3 \tag{33}$$

We now have

$$\boldsymbol{A} = \boldsymbol{a} + [\boldsymbol{a}\,\boldsymbol{w} - \boldsymbol{w}\,\boldsymbol{a}] + \cdots$$

in leading order. By calculation

$$\begin{aligned}[\boldsymbol{a}\,\boldsymbol{w} - \boldsymbol{w}\,\boldsymbol{a}] = {}& \;\;2(-w^3a^2 + w^2a^3)\,\boldsymbol{e}_1 \\ &+ 2(+w^3a^1 - w^1a^3)\,\boldsymbol{e}_2 \\ &+ 2(+w^2a^1 - w^1a^2)\,\boldsymbol{e}_3\end{aligned} \tag{34}$$

so in matrix representation we have

$$\mathbb{U} = \mathbb{I} + \mathbb{W} \quad \text{where} \quad \mathbb{W} \equiv 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -w^3 & +w^2 \\ 0 & +w^3 & 0 & -w^1 \\ 0 & +w^2 & -w^1 & 0 \end{pmatrix} \tag{35}$$

Notice now that the modulus of $\boldsymbol{a}$ can be written

$$|\,\boldsymbol{a}\,| = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}^{\!\mathsf{T}} \mathbb{G} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad \text{with} \quad \mathbb{G} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

and that modulus presevation entails $\mathbb{U}^{\mathsf{T}}\mathbb{G}\mathbb{U} = \mathbb{G}$ whence (in leading order) $\mathbb{W}^{\mathsf{T}}\mathbb{G} + \mathbb{G}\mathbb{W} = \mathbb{O}$ which can be written $\mathbb{W}^{\mathsf{T}} = -\mathbb{G}\mathbb{W}\mathbb{G}^{-1}$ or again

$$(\mathbb{G}\,\mathbb{W})^{\mathsf{T}} = -(\mathbb{G}\,\mathbb{W}) \tag{37}$$

We verify that the matrices $\mathbb{W}$ and $\mathbb{G}$ defined above do in fact satisfy that "$\mathbb{G}$-antisymmetry" condition. We write

$$\mathbb{W} = 2w^1\mathbb{J}_1 + 2w^2\mathbb{J}_2 + 2w^3\mathbb{J}_3 \tag{38}$$

and observe that the matrices

$$\mathbb{J}_1 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},\; \mathbb{J}_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix},\; \mathbb{J}_3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[1] The plan is to construct finite similarity transformations by iteration of such infinitesimal transformations.

thus defined are—though not closed multiplicatively (therefore *not* candidates to provide matrix representatives of the algebraic objects $\boldsymbol{e}_1$, $\boldsymbol{e}_3$, $\boldsymbol{e}_3$)—closed under commutation:

$$\left.\begin{aligned} \mathbb{J}_1\mathbb{J}_2 - \mathbb{J}_2\mathbb{J}_1 &= -\mathbb{J}_3 \\ \mathbb{J}_2\mathbb{J}_3 - \mathbb{J}_3\mathbb{J}_2 &= +\mathbb{J}_1 \\ \mathbb{J}_3\mathbb{J}_1 - \mathbb{J}_1\mathbb{J}_3 &= +\mathbb{J}_2 \end{aligned}\right\} \tag{39}$$

Except for the goofy signs these commutation relations resemble those we associate with the generators of $O(3)$, the 3-dimensional rotation group. Entrusting all computational work to *Mathematica*, we discover that

$$\begin{aligned} &\lambda^4 - \lambda^2 = 0 \text{ is the characteristic equation of both } \mathbb{J}_1 \text{ and } \mathbb{J}_2 \\ &\lambda^4 + \lambda^2 = 0 \text{ is the characteristic equation of } \mathbb{J}_3 \end{aligned}$$

so

$$\begin{aligned} &\big\{+1, -1, 0, 0\big\} \text{ are the eigenvalues of both } \mathbb{J}_1 \text{ and } \mathbb{J}_2 \\ &\big\{+i, -i, 0, 0\big\} \text{ are the eigenvalues of } \mathbb{J}_3 \end{aligned}$$

We verify that each of the $\mathbb{J}$-matrices satisfies its own characteristic equation (as the Hamilton-Jacobi theorem requires), and discover that in fact

$$\begin{aligned} &\mathbb{J}_1 \text{ and } \mathbb{J}_2 \text{ satisfy the reduced characteristic equation } \mathbb{J}^3 - \mathbb{J} = \mathbb{O} \\ &\mathbb{J}_3 \text{ satisfies the reduced characteristic equation } \mathbb{J}^3 + \mathbb{J} = \mathbb{O} \end{aligned}$$

More to the point: the characteristic equation of $\mathbb{W}$ reads[2]

$$\lambda^4 - 4(w_1^2 + w_2^2 - w_3^2)\,\lambda^2 = 0$$

which yields eigenvalues

$$\big\{+2\sqrt{w_1^2 + w_2^2 - w_3^2}\,, -2\sqrt{w_1^2 + w_2^2 - w_3^2}\,, 0, 0\big\}$$

and the reduced Hamilton-Jacobi statement

$$\mathbb{W}^3 - 4(w_1^2 + w_2^2 - w_3^2)\,\mathbb{W} = \mathbb{O} \tag{40}$$

Turning now from the infinitesimal to the finite aspects of the theory, let the infinitesimal w-triplet be written

$$\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} = \tfrac{1}{N}\,\theta \begin{pmatrix} k^1 \\ k^2 \\ k^3 \end{pmatrix} \quad \text{with} \quad \begin{cases} k_1^2 + k_1^2 + k_1^2 = +1, \text{ else} \\ k_1^2 + k_1^2 + k_1^2 = 0, \text{ else} \\ k_1^2 + k_1^2 + k_1^2 = -1 \end{cases}$$

where our obligation to distinguish three cases arises from the indefinitness of

[2] Here—and occasionally hereafter—I allow myself to write (for example) w_1^2 where I should more properly write w^1w^1 or $(w^1)^2$.

the metric matrix $\mathbb{G}$. Iteration of (35) then gives[3]

$$\begin{aligned}\mathbb{U}^N &= \Big[\mathbb{I} + \tfrac{1}{N}2\theta\big\{k^1\mathbb{J}_1 + k^2\mathbb{J}_2 + k^3\mathbb{J}_3\big\}\Big]^N\\ &\downarrow\\ \mathbb{U}(\theta;\boldsymbol{k}) &= \exp\Big[2\theta\big\{k^1\mathbb{J}_1 + k^2\mathbb{J}_2 + k^3\mathbb{J}_3\big\}\Big] \quad \text{as} \quad N\uparrow\infty \\ &\equiv e^{2\theta\mathbb{K}}\end{aligned} \tag{41}$$

We have now in hand enough algebraic information to develop and interpret the action of the transformation matrix $e^{2\theta\mathbb{K}}$. The technique is pretty,[4] but its details need not concern us at the moment. It is sufficient to notice that the $\mathbb{G}$-antisymmetry of $\mathbb{K}$ forces $\mathbb{U} \equiv e^{2\theta\mathbb{K}}$ to be $\mathbb{G}$-orthogonal:

$$\mathbb{G}^{-1}\mathbb{K}^{\mathsf{T}}\mathbb{G} = -\mathbb{K} \quad\Longrightarrow\quad \mathbb{G}^{-1}\mathbb{U}^{\mathsf{T}}\mathbb{G} = \mathbb{U}^{-1} \tag{42}$$

And that *Mathematica* today stands ready to do (in, typically, 0.0166 seconds!) all the work. Commands of the form `MatrixExp[`*matrix*`]//MatrixForm` yielded the following illuminating results:

$$e^{2\theta\mathbb{J}_1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cosh 2\theta & -\sinh 2\theta\\ 0 & 0 & -\sinh 2\theta & \cosh 2\theta \end{pmatrix}$$

$$e^{2\theta\mathbb{J}_2} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cosh 2\theta & 0 & \sinh 2\theta\\ 0 & 0 & 1 & 0\\ 0 & \sinh 2\theta & 0 & \cosh 2\theta \end{pmatrix}$$

$$e^{2\theta\mathbb{J}_3} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos 2\theta & -\sin 2\theta & 0\\ 0 & \sin 2\theta & \cos 2\theta & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{2\theta(\mathbb{J}_2+\mathbb{J}_3)} = \mathbb{I} + 2\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -\theta^1 & \theta^1\\ 0 & \theta^1 & -\theta^2 & \theta^2\\ 0 & \theta^1 & -\theta^2 & \theta^2 \end{pmatrix}$$

Notice in the connection with

- the first example that $-1^2 - 0^2 + 0^2 = -1$
- the second example that $-0^2 - 1^2 + 0^2 = -1$
- the third example that $-0^2 - 0^2 + 1^2 = +1$
- the fourth example that $-0^2 - 1^2 + 1^2 = 0$ and the series terminates.

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[3] I am running out of letters and fonts. In the notation advanced at (41) what I formerly called $\mathbb{U}$ would now be denoted $\mathbb{U}(\delta\theta;\boldsymbol{k})$.

[4] For my most recent discussion of this subject, and references to earlier treatments, see §4 in "Extrapolated interpolation theory" (1997).

The first example describes what is, in effect, a *Lorentzian boost* along the negative 2-axis (the 3-axis being identified with the "time" axis); the second describes a boost along the positive 1-axis; the third describes a *rotation* in the $(1,2)$-plane. The final example describes a transformation that is degenerate: its action is certainly describable, but I will not linger to do so.

The 2-factor in the exponent at (41) is a story in itself: it is most familiar as the source of the double-valuedness of the spinor representations of $O(3)$, but that is only one of its manifestations: it arises in *all* such contexts.

It remains only to construct a **matrix representation** of our Clifford algebra. Here—in the absence of a deductive procedure—I am obliged to proceed by improvisation, by modification of rabbits pulled from Pauli's hat. The Pauli matrices are standardly defined[5]

$$\boldsymbol{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{43}$$

though some authors adopt similarity-equivalent alternatives to those matrices. The Pauli matrices are traceless, Hermitian, and satisfy the relations

$$\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_2^2 = \mathbb{I} \tag{44.1}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 &= i\boldsymbol{\sigma}_3 = -\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 &= i\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 &= i\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 \end{aligned}\right\} \tag{44.2}$$

We are inspired to introduce

$$\left.\begin{aligned} \mathbb{e}_1 \equiv \boldsymbol{\sigma}_1 \quad &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \mathbb{e}_2 \equiv \boldsymbol{\sigma}_2 \quad &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \mathbb{e}_3 \equiv \mathbb{e}_1\mathbb{e}_2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{aligned}\right\} \tag{45}$$

which evidently/demonstrably satisfy

$$\mathbb{e}_1^2 = \mathbb{e}_2^2 = \mathbb{I}\,, \quad \mathbb{e}_3^2 = -\mathbb{I} \tag{46.1}$$

$$\left.\begin{aligned} \mathbb{e}_1\mathbb{e}_2 &= \mathbb{e}_3 = -\mathbb{e}_2\mathbb{e}_1 \\ \mathbb{e}_2\mathbb{e}_3 &= -\mathbb{e}_1 = -\mathbb{e}_3\mathbb{e}_2 \\ \mathbb{e}_3\mathbb{e}_1 &= -\mathbb{e}_2 = -\mathbb{e}_3\mathbb{e}_2 \end{aligned}\right\} \tag{46.2}$$

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[5] See David Griffiths, *Introduction to Quantum Mechanics* (1995), page 156. See also page 2 in Chapter 1 of my *Advanced Quantum Topics* (2000).

which are just what we need. If we introduce

$$\begin{aligned}\mathbb{a} &= a^0\,\mathbb{I} + a^1\mathbb{e}_1 + a^2\mathbb{e}_2 + a^3\mathbb{e}_3\\ \mathbb{b} &= b^0\,\mathbb{I} + b^1\mathbb{e}_1 + b^2\mathbb{e}_2 + b^3\mathbb{e}_3\end{aligned}$$

and with *Mathematica*'s assistance compute $\mathbb{a}\mathbb{b}$ we obtain a result which is precise agreement with (24). In

$$\boldsymbol{a} = a^0\mathbf{I} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3 \;\longleftrightarrow\; \mathbb{a} = a^0\,\mathbb{I} + a^1\mathbb{e}_1 + a^2\mathbb{e}_2 + a^3\mathbb{e}_3 \tag{47}$$

we have, therefore, a complex 2×2 matrix representation of the Clifford algebra that was called into being at (4). We note with interest that

$$\det\mathbb{a} = a^0a^0 - a^1a^1 - a^2a^2 + a^3a^3 = \text{modulus }|\boldsymbol{a}| \tag{47}$$

At (14) we encountered a 2×2 real matrix representation of i. Acting now on a hunch, we make substitutions

$$1 \mapsto \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad 0 \mapsto \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \quad i \mapsto \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

into the equations (45) that defined the $\mathbb{e}$-matrices and obtain

$$\left.\begin{aligned}\mathbb{E}_1 &\equiv \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\\ \mathbb{E}_2 &\equiv \begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}\\ \mathbb{E}_3 &\equiv \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\end{aligned}\right\} \tag{49}$$

We are informed by *Mathematica* that these real matrices satisfy relations identical to the relations (46) satisfied by the complex $\mathbb{e}$-matrices. In

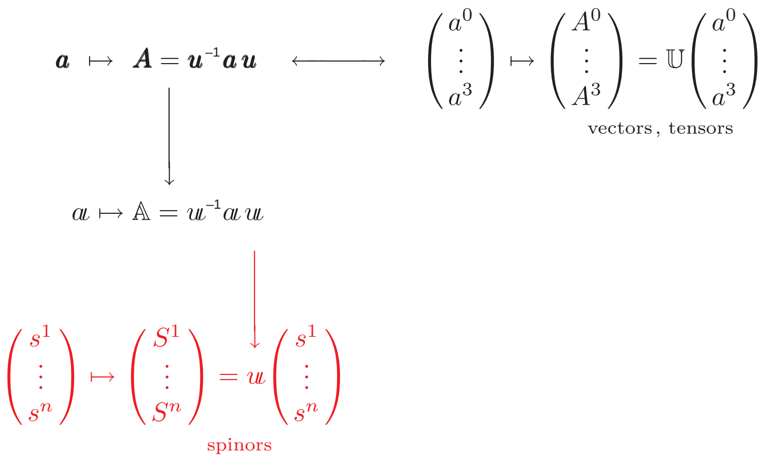
$$\boldsymbol{a} = a^0\mathbf{I} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3 \;\longleftrightarrow\; \mathbb{A} = a^0\,\mathbb{I} + a^1\mathbb{E}_1 + a^2\mathbb{E}_2 + a^3\mathbb{E}_3 \tag{50}$$

we have, therefore, a real 4×4 matrix representation of the Clifford algebra of order 2, in which connection we observe that

$$\det\mathbb{A} = (a^0a^0 - a^1a^1 - a^2a^2 + a^3a^3)^2 = (\text{modulus }|\boldsymbol{a}|)^2 \tag{51}$$

It is important not to confuse the 4-dimensionality of recent discussion with the 4-dimensionality that laid claim to our attention at (32). Our recent work has placed us in position to display a 4×4 real representation (alternatively a complex 2×2 complex representation) of (31), whereas the work of pages 6–9 was concerned with the representation of (32). Those two transformation principles are of quite different design... yet—and this is the point—manage to subject the numbers $\{a^0, a^1, a^2, a^3\}$ to the *same adventure*.

It would be a story well worth the telling if it ended there. But it doesn't. We have busied ourselves thus far with spelling out the specific meaning—or at least the meaning assumed within $\mathcal{C}_2$, the Clifford algebra of order 2—of the upper (black) portion of the following figure, but find ourselves in position now



to descend to its lower (red) left corner, where we find a transformation law that—though $\mathbb{U}$, $\boldsymbol{u}$ and $\mathbb{u}$ all encode the same data—is distinct from the transformation law seen at upper right: while the numbers $\{a^0, a^1, a^2, a^3\}$ go adventuring so, in their wake, do the numbers $\{s^1, \ldots, s^n\}$, but in their own distinctive way. What began at (20) as an attempt to construct a formal square root of (a 2-dimensional instance of) the familiar inner product has resulted finally in what might, in a manner of speaking, be called the *square root of vector algebra itself!*

3. Quaternions: a digression. From the Pauli matrices (43) construct the traceless *anti*hermitian matrices $\mathbb{h}_j \equiv (1/i)\boldsymbol{\sigma}_j$: $j = 1, 2, 3$. Working from (44) we have

$$\mathbb{h}_1^2 = \mathbb{h}_2^2 = \mathbb{h}_3^2 = -\mathbb{I} \tag{52.1}$$

$$\left.\begin{aligned} \mathbb{h}_1\mathbb{h}_2 &= \mathbb{h}_3 = -\mathbb{h}_2\mathbb{h}_1 \\ \mathbb{h}_2\mathbb{h}_3 &= \mathbb{h}_1 = -\mathbb{h}_3\mathbb{h}_2 \\ \mathbb{h}_3\mathbb{h}_1 &= \mathbb{h}_2 = -\mathbb{h}_1\mathbb{h}_3 \end{aligned}\right\} \tag{52.2}$$

which have the attractive property that the i-factors present in (44) have now disappeared. The i-factors were of no concern to Pauli, but their presence would be unwelcome if our objective were to construct a generalization of complex algebra. To the latter end, posit the existence of abstract objects $\boldsymbol{h}_1$, $\boldsymbol{h}_2$, $\boldsymbol{h}_3$ that satisfy the relations (52), and after notational adjustments

$$\begin{aligned}\boldsymbol{h}_1 &\text{ becomes } \boldsymbol{i}\\ \boldsymbol{h}_2 &\text{ becomes } \boldsymbol{j}\\ \boldsymbol{h}_3 &\text{ becomes } \boldsymbol{k}\end{aligned}$$

obtain

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\mathsf{I} \tag{53.1}$$

$$\left.\begin{aligned}\boldsymbol{i}\boldsymbol{j} = \boldsymbol{k} &= -\boldsymbol{j}\boldsymbol{i}\\ \boldsymbol{j}\boldsymbol{k} = \boldsymbol{i} &= -\boldsymbol{k}\boldsymbol{j}\\ \boldsymbol{k}\boldsymbol{i} = \boldsymbol{j} &= -\boldsymbol{i}\boldsymbol{k}\end{aligned}\right\} \tag{53.2}$$

These are equations that, after a long period of frustrated thought, occurred to William Rowan Hamilton in a flash on Monday, 16 October 1843, as he strolled with his wife across Brougham Bridge, in Dublin, on his way to a meeting of the Royal Irish Academy. To establish his priority he scratched equations (53) onto the bridge rail.[6] That very afternoon he announced to the Academy his intention to read—and on Monday, November 13th did read—the first of his many papers on what he by then called the "theory of quaternions." Hamilton's motivation, which had a very strong but obscurely idiosyncratic philosophical component, is difficult for modern readers to grasp.[7] But his accomplishment is easy to grasp—easier for us, no doubt, than it was for Hamilton: he had introduced into vocabulary of mathematics the concept of **non-commutivity**. He had planted one of the seeds (Hermann Grassmann, at about the same time, planted another) from which the theory off algebras in general, and Clifford algebras in particular, were soon to sprout.

It follows from (53) that if

$$\begin{aligned}\boldsymbol{a} &= a^0\mathsf{I} + a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k}\\ \boldsymbol{b} &= b^0\mathsf{I} + b^1\boldsymbol{i} + b^2\boldsymbol{j} + b^3\boldsymbol{k}\end{aligned}$$

[6] In a letter written in 1865, shortly before his death, Hamilton claimed to have written $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathsf{I}$, from which equations (53.2) can be recovered as corollaries. It seems doubtful that Hamilton was so sophisticated at such an early point in his work, but perhaps he was: the physical evidence has long since vanished.

[7] See the discussion in Chapters 6 & 7 of T. L. Hankins, *Sir William Rowan Hamilton* (1980) and Chapter 2 of M. J. Crowe, *A History of Vector Analysis*: *The Evolution of the Idea of a Vectorial System* (1967).

are quaternions then their product[8] can be described

$$\begin{aligned}\boldsymbol{ab} = (a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3)\,\mathsf{I} &+ (a^0b^1 + a^1b^0 + a^2b^3 - a^3b^2)\,\boldsymbol{i}\\ &+ (a^0b^2 + a^2b^0 + a^3b^1 - a^1b^3)\,\boldsymbol{j}\\ &+ (a^0b^3 + a^3b^0 + a^1b^2 - a^2b^1)\,\boldsymbol{k}\end{aligned} \tag{54}$$

It follows that if we define

$$\overline{\boldsymbol{a}} \equiv a^0\mathsf{I} - a^1\boldsymbol{i} - a^2\boldsymbol{j} - a^3\boldsymbol{k} \tag{55}$$

then

$$\begin{aligned}\boldsymbol{a}\overline{\boldsymbol{a}} &= (a^0a^0 + a^1a^1 + a^2a^2 + a^3a^3)\,\mathsf{I}\\ &= \overline{\boldsymbol{a}}\boldsymbol{a}\end{aligned} \tag{56}$$

and that $\boldsymbol{a}^{-1}$ can be described

$$\boldsymbol{a}^{-1} = \frac{\overline{\boldsymbol{a}}}{a^0a^0 + a^1a^1 + a^2a^2 + a^3a^3} \tag{57}$$

which (on the assumption that the a's are real) exists except in the case $\boldsymbol{a} = \boldsymbol{0}$. We agree to call

$$|\boldsymbol{a}| \equiv \sqrt{a^0a^0 + a^1a^1 + a^2a^2 + a^3a^3} \geqslant 0 \tag{58}$$

the "modulus" of the quaternion $\boldsymbol{a}$ and establish by calculation that

$$|\boldsymbol{a}_1 \cdot \boldsymbol{a}_2| = |\boldsymbol{a}_1| \cdot |\boldsymbol{a}_2| \tag{59}$$

We stand now in need of some sharpened terminology: let the coefficient a^0 of I in the development of the quaternion $\boldsymbol{a}$ be called the "spur" of $\boldsymbol{a}$:

$$\mathrm{sp}(a^0\mathsf{I} + a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k}) \equiv a^0 \tag{60}$$

It follows from (54) that

$$\mathrm{sp}(\boldsymbol{ab}) = \mathrm{sp}(\boldsymbol{ba}) \tag{61}$$

[8] It was Hamilton who, in order to drive the evil i's from the temple, had been the first to propose that complex numbers be construed to be *ordered pairs* of real numbers, subject to the multiplication law

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

It was, I presume, the 3-dimensionality of physical space that inspired his interest in *ordered triplets*. He recalled late in life, in a letter to his eldest son, that "every morning…on my coming down to breakfast, [you and your brother] used to ask me, "Well, Papa, can you *multiply* triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only *add* and subtract them." (My source here has been Crowe's page 29.) Hamilton cannot have anticipated that his triplets would have to be embedded within quartets, or that his banished i would return with two even more spooky friends.

It is in view of the fact that, in matrix theory,

$$\mathrm{tr}(\mathbb{A}\mathbb{B}) = \mathrm{tr}(\mathbb{B}\mathbb{A})$$

and because I want to preserve the "trace" for matrix-theoretic applications . . . that I have pressed into quaternionic service its German equivalent. To that terminology I add now more: we agree to say of a quaternion $\boldsymbol{a}$ that it is "pure" if and only if its spur vanishes. In short:

$$\boldsymbol{a},\ \text{if "pure," has the form } a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k}$$

and when $\boldsymbol{a}$ is *not* pure we will call $a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k}$ its "pure part" (just as we speak of the "imaginary part" of a complex number).

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are pure then, by (54), we have

$$\begin{aligned} \boldsymbol{x}\boldsymbol{y} = -(x^1y^1 + x^2y^2 + x^3y^3)\,\mathbf{I} &+ (x^2y^3 - x^3y^2)\,\boldsymbol{i} \\ &+ (x^3y^1 - x^1y^3)\,\boldsymbol{j} \\ &+ (x^1y^2 - x^2y^1)\,\boldsymbol{k} \end{aligned} \tag{62}$$

Look now to the quaternionic similarity transformation

$$\boldsymbol{a} \longmapsto \boldsymbol{A} = \boldsymbol{u}^{-1}\boldsymbol{a}\,\boldsymbol{u} \tag{63}$$

where one can, without loss of generality, assume $\boldsymbol{u}$ to be unimodular. Such transformations are, by (59) modulus-preserving. And they are, by (61), also spur-preserving:

$$A^0 = a^0 \quad : \quad \text{all } \boldsymbol{u} \tag{64}$$

It follows that we might as well assume from the outset that $\boldsymbol{a}$ is pure. This we do, and emphasize by notational adjustment: in place of (63) we write

$$(x^1\boldsymbol{i} + x^2\boldsymbol{j} + x^3\boldsymbol{k}) \longmapsto (X^1\boldsymbol{i} + X^2\boldsymbol{j} + X^3\boldsymbol{k}) = \boldsymbol{u}^{-1}(x^1\boldsymbol{i} + x^2\boldsymbol{j} + x^3\boldsymbol{k})\,\boldsymbol{u} \tag{65}$$

From $X^1X^1 + X^2X^2 + X^3X^3 = x^1x^1 + x^2x^2 + x^3x^3$ we conclude that such transformations admit of the alternative description

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \longmapsto \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} = \mathbb{R}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{66}$$

where $\mathbb{R}$ is a 3×3 rotation matrix. I will not proceed farther down this road: it is a road too well traveled . . . though it leads pretty things, valuable things.

Retreating to the $\mathbb{h}$-matrices that at (52) marked our point of departure, we have already in hand a 2×2 complex matrix representation of Hamilton's quaternion algebra:

$$\left.\begin{aligned} \boldsymbol{i} \;&\longleftrightarrow\; \mathbb{h}_1 = -i\,\boldsymbol{\varpi}_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\ \boldsymbol{j} \;&\longleftrightarrow\; \mathbb{h}_2 = -i\,\boldsymbol{\varpi}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \boldsymbol{k} \;&\longleftrightarrow\; \mathbb{h}_3 = -i\,\boldsymbol{\varpi}_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \end{aligned}\right\} \tag{67}$$

The representative of $\boldsymbol{a} = a^0\mathbf{I} + a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k}$ therefore reads

$$\mathbb{A} = \begin{pmatrix} a^0 - ia^3 & -a^2 - ia^1 \\ a^2 - ia^1 & a^0 + ia^3 \end{pmatrix}$$

and we have

$$\det \mathbb{A} = a^0a^0 + a^1a^1 + a^2a^2 + a^3a^3 = |\boldsymbol{a}|^2 \tag{68.1}$$

$$\operatorname{tr} \mathbb{A} = 2a^0 = 2\,\mathrm{sp}(\boldsymbol{a}) \tag{68.2}$$

which render explicit the relationships between the quaternionic "modulus" and "spur" and their matrix-theoretic counterparts. "Representation theory" leads also to good things, but here again they are things too familiar to require explicit review on this occasion. It is, by the way, my impression that we touch here upon an aspect of his theory that Hamilton—who worked when the theory of matrices was still in its infancy—did himself *not* explore.[9]

To summarize: the relationship between Hamilton's quaternion algebra $\mathcal{Q}$ and the simplest Clifford algebra $\mathcal{C}_2$ is intimate, but curiously skew.

Hamilton introduces a triple of algebraic objects $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, to which he assigns co-equal status. The object $x_1^2+x_2^2+x_3^2$ emerges naturally but incidentally from his theory: it is not an object to which generative significance is assigned.

Clifford does assign generative significance to $x_1^2 + x_2^2$. He is led to an algebraic construct in which $\boldsymbol{e}_1 = i\boldsymbol{i}$ and $\boldsymbol{e}_2 = i\boldsymbol{j}$ play the role of *generators* and into which $\boldsymbol{e}_3 \equiv \boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{k}$ is introduced simply to achieve algebraic closure. In the fully-elaborated theory it is not $x_1^2 + x_2^2 + x_3^2$ but $-x_1^2 - x_2^2 + x_3^2$ that acquires the status of a natural object.

Hamilton devoted the last twenty-two years of his life to the development and promotion of the theory of quaternions, to which he "was inclined to imbue with cosmic significance."[10] British and American mathematicians and

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[9] I do not have access to Hamilton's *Lectures on Quaternions* (1853) or to his posthumous *Elements of Quaternions* (1866), which should be consulted in this regard. The theory of matrices originates in work published by Arthur Cayley in 1858.

[10] The phrase is Carl Boyer's: see page 625 in his *A History of Mathematics* (1968).

mathematical physicists, during the closing decades of the 19^{th} Century, tended fairly generally to be quaternionists (or at least to pay lip service to the new religion), though some became outspoken critics of the trend, and others were content to entertain various shades of bemused indifference.[11] Today it is universally recognized that the *invention* of quaternions (which is to say: of non-commutivity) was a seminal event, though the quaternion algebra itself has become a relatively insignificant detail within a vast subject. Such importance as it does enjoy is due more to the work of Pauli than of Hamilton. And yet, echos of the former cult status of quaternion algebra persist to this day: Google responds with more than 59,000 items to the key-word "quaternion," and much of that work appears on cursory inspection to be fairly self-indulgent, having little to do with anything.

The situation with regard to Clifford's invention could hardly be more different. The birthplace of "Clifford algebra" is difficult to discover within Clifford's *Mathematical Papers*: the essential thought was presented as but one idea among the bewilderingly many, an incidental bi-product of his interest in the work of Grassmann... and it was certainly not an idea he chose to cultivate, to promote. That work fell to others, decades later. Today, Clifford is a cult figure, and his algebra an object of worship. A journal *Advances in Applied Clifford Algebras* exists, International Clifford Algebra Conferences are held, Google responds with more than 45,000 items to the key-word "Clifford algebra." I have learned to keep this work at arm's length not because it is frivolous (though some of it certainly is) but because it tends to be seductive, the rewards disproportionate to the investment The present project represents a departure from that personal policy.

Hamilton's vision (shared most vocally/influentially by Peter Guthrie Tait) of a fully "quaternionized physics" was ultimately subverted by a combiniation of circumstances, among them

- the accumulated weight of the formalism
- the discovery of simpler, more direct ways to manage multi-dimensional objects
- the discovery that physics has need sometimes of algebraic structures more complicated than (or at least alternative to) quaternions

Under the second head we might cite the invention (beginning in the 1880's) of tensor analysis, and the work of

Gibbs & Heaviside who, in the early 1880's, independently invented the formalism known today as **vector algebra & analysis**. Gibbs, though familiar with Hamilton, claimed Grassmann as his principal influence, while Heaviside (who probably never heard of Grassmann) worked in direct reaction to Hamilton. It was the idea of each to squeeze the juice from quaternions and discard the

[11] I am thinking here especially of Maxwell... whose passing mention (in his *Treatise*) of quaternions did, however, lead both Gibbs and Heaviside to take up—only to abandon—the subject.

rind. The implications of that idea were clearly spelled out in J. W. Gibbs & E. B. Wilson's *Vector Analysis* (1901), which was based on class notes developed by Gibbs during the 1880's and 1890's, and was the influential first textbook in the field. Gibbs (like Heaviside) considered $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ to refer to objects no more mysterious than <u>orthogonal unit vectors in 3-space</u>. "Pure quaternions"

$$\begin{aligned}\boldsymbol{x} &= x^1\boldsymbol{i} + x^2\boldsymbol{j} + x^3\boldsymbol{k}\\ \boldsymbol{y} &= y^1\boldsymbol{i} + y^2\boldsymbol{j} + y^3\boldsymbol{k}\end{aligned}$$

become by this interpretation simple 3-vectors. Drawing inspiration from (62), Gibbs defined two distinct kinds of "product":

$$\left.\begin{aligned}\text{number-valued } \textit{dot product } \boldsymbol{x}\cdot\boldsymbol{y} &\equiv x^1y^1 + x^2y^2 + x^3y^3\\ \text{vector-valued } \textit{cross product } \boldsymbol{x}\times\boldsymbol{y} &\equiv \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ x^1 & x^2 & x^3\\ y^1 & y^2 & y^3\end{vmatrix}\end{aligned}\right\} \tag{69}$$

And he <u>abandoned all thought of "dividing a vector by a vector</u>," perhaps because the meaning Hamilton would so proudly assign to

$$\boldsymbol{x}\boldsymbol{y}^{-1} = -\frac{1}{\boldsymbol{y}\cdot\boldsymbol{y}}\Big\{(\boldsymbol{x}\cdot\boldsymbol{y})\mathsf{I} + (\boldsymbol{x}\times\boldsymbol{y})\Big\} \tag{70}$$

would be a hybrid object, vectorially meaningless unless either $\boldsymbol{x}\cdot\boldsymbol{y} = 0$ or $\boldsymbol{x}\times\boldsymbol{y} = \mathbf{0}$. It is remarkable, when you think about it, how successful are the applications of vector analysis to solid geometry and 3-dimensional physics, given that vector analysis provides *no concept of vector division*.[12]

4. Second order Clifford algebra with general metric. We turn now to study of the implications of writing, in place of (20),

$$(g_{ij}x^ix^j) = (x^i\boldsymbol{e}_i)^2 \quad : \quad \text{all } x^i \tag{71}$$

where

$$\mathbb{g} \equiv \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}$$

is understood to be real, symmetric and non-singular ($g \equiv \det\mathbb{g} \neq 0$) but we will not impose the requirement that $\mathbb{g}$ be positive-definite ($g > 0$). It has been my experience (in other, more complicated, contexts) that metric generalization

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \longmapsto \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}$$

[12] For extended discussion of the mathematical developments that historically radiated from Hamilton's invention, and more detailed references, see "Theories of Maxwellian design" (1998).

tends to complicate life at the outset, but that the added effort is worthwhile in the longrun, for it exposes important distinctions that otherwise remain invisible.

Immediately

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2g_{ij}\mathbf{I} \tag{72}$$

The antisymmetry condition (22.2) is now lost, though we have in its place the antisymmetry of

$$\boldsymbol{e}_{ij} \equiv \boldsymbol{e}_i\boldsymbol{e}_j - g_{ij}\mathbf{I} \quad : \quad \boldsymbol{e}_{ij} = -\boldsymbol{e}_{ji} \tag{73}$$

It becomes therefore natural to close the $\{\boldsymbol{e}_i, \boldsymbol{e}_j\}$-generated algebra with the introduction of

$$\begin{aligned} \boldsymbol{f} &\equiv \tfrac{1}{2}\varepsilon^{ij}\boldsymbol{e}_{ij} && (74.1)\\ &= \boldsymbol{e}_{12} \\ &= \quad \boldsymbol{e}_1\boldsymbol{e}_2 - g_{12}\mathbf{I} && (74.2)\\ &= -\boldsymbol{e}_2\boldsymbol{e}_1 + g_{21}\mathbf{I} && (74.3) \end{aligned}$$

From (72)–(74) we extract the primative products

$$\begin{array}{lll} \boldsymbol{e}_1\boldsymbol{e}_1 = g_{11}\mathbf{I} & \boldsymbol{e}_1\boldsymbol{e}_2 = g_{12}\mathbf{I} + \boldsymbol{f} & \boldsymbol{e}_1\boldsymbol{f} = g_{11}\boldsymbol{e}_2 - g_{12}\boldsymbol{e}_1 \\ \boldsymbol{e}_2\boldsymbol{e}_1 = g_{21}\mathbf{I} - \boldsymbol{f} & \boldsymbol{e}_2\boldsymbol{e}_2 = g_{22}\mathbf{I} & \boldsymbol{e}_2\boldsymbol{f} = g_{21}\boldsymbol{e}_2 - g_{22}\boldsymbol{e}_1 \\ \boldsymbol{f}\,\boldsymbol{e}_1 = g_{21}\boldsymbol{e}_1 - g_{11}\boldsymbol{e}_2 & \boldsymbol{f}\,\boldsymbol{e}_2 = g_{22}\boldsymbol{e}_1 - g_{12}\boldsymbol{e}_2 & \boldsymbol{f}\,\boldsymbol{f} = -g\mathbf{I} \end{array}$$

where in the final equation $g \equiv g_{11}g_{22} - g_{12}g_{21} = \det \mathbb{g}$. Therefore and equivalently: if

$$\begin{aligned} \boldsymbol{c} &\equiv s\,\mathbf{I} + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\boldsymbol{f} \\ \boldsymbol{C} &\equiv S\,\mathbf{I} + V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + P\boldsymbol{f} \end{aligned}$$

are arbitrary Clifford numbers then

$$\begin{aligned} \boldsymbol{c}\boldsymbol{C} = &s\big[S\,\mathbf{I} + V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + P\boldsymbol{f}\big] \\ &+ v^1\big[S\boldsymbol{e}_1 + V^1(g_{11}\mathbf{I}) + V^2(g_{12}\mathbf{I} + \boldsymbol{f}) + P(g_{11}\boldsymbol{e}_2 - g_{12}\boldsymbol{e}_1)\big] \\ &+ v^2\big[S\boldsymbol{e}_2 + V^1(g_{21}\mathbf{I} - \boldsymbol{f}) + V^2(g_{22}\mathbf{I}) + P(g_{21}\boldsymbol{e}_2 - g_{22}\boldsymbol{e}_1)\big] \\ &+ \ p\,\big[S\boldsymbol{f} + V^1(g_{21}\boldsymbol{e}_1 - g_{11}\boldsymbol{e}_2) + V^2(g_{22}\boldsymbol{e}_1 - g_{12}\boldsymbol{e}_2) + P(-g\mathbf{I})\big] \\ = &\,\mathbf{I}\,\big[sS + (v^1V_1 + v^2V_2) - gpP\big] \\ &+ \boldsymbol{e}_1\big[sV^1 + v^1S - (v^1g_{12} + v^2g_{22})P + p(g_{21}V^1 + g_{22}V^2)\big] \\ &+ \boldsymbol{e}_2\big[sV^2 + v^2S + (v^1g_{11} + v^2g_{21})P - p(g_{11}V^1 + g_{12}V^2)\big] \\ &+ \boldsymbol{f}\,\big[sP + v^1V^2 - v^2V^1 + pS\big] \\ = &\,\mathbf{I}\,\big[sS + v^nV_n - gpP\big] \\ &+ \boldsymbol{e}_1\big[sV^1 + v^1S - v_2P + pV_2\big] \\ &+ \boldsymbol{e}_2\big[sV^2 + v^2S + v_1P - pV_1\big] \\ &+ \boldsymbol{f}\,\big[sP + v^1V^2 - v^2V^1 + pS\big] \end{aligned} \tag{75.1}$$

Here I have used g_{mn} to lower indices, in the manner standard to tensor analysis: $v_m \equiv g_{m1}v^1 + g_{m2}v^2 \equiv g_{mn}v^n$ and $V_m \equiv g_{mn}V^n$.

> **IMPORTANT REMARK:** I should emphasize that the contravariant Levi-Civita symbol ε^{ij} encountered at (74.1) can *in every coordinate system* be described
>
> $$\varepsilon^{ij} = \begin{cases} \text{sgn}\begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix} & : \quad i \;\&\; j \text{ distinct} \\ 0 & : \quad \text{otherwise} \end{cases}$$
>
> if and only if it is understood to transform as a density of weight $W = +1$. That same weight then attaches automatically to its covariant companion
>
> $$\varepsilon_{ij} = g_{im}g_{jn}\varepsilon^{mn}$$
>
> Note, however, that while the numerical values of ε^{ij} range on $\{-1, 0, +1\}$ those of ε_{ij} range on $\{-g, 0, +g\}$ Similarly, the defining statement
>
> $$\epsilon_{ij} = \begin{cases} \text{sgn}\begin{pmatrix} i & j \\ 1 & 2 \end{pmatrix} & : \quad i \;\&\; j \text{ distinct} \\ 0 & : \quad \text{otherwise} \end{cases}$$
>
> holds in every coordinate system iff ϵ_{ij} is understood to transform as a density of weight $W = -1$. One has
>
> $$\varepsilon_{ij} = g\epsilon_{ij}$$
>
> and achieves consistency with the observation that
>
> $$g \equiv \det \mathbb{g} \text{ transforms as a scalar density of weight } W = +2$$
>
> I honor the notational conventions adopted in "Electrodynamical applications of the exterior calculus" (1996). See pages 7 & 9 for more detailed discussion of the points at issue.

The preceding remarks place us in position to consider the *transformational properties* of the results in hand (which, so long as the metric was required to be Euclidean, we were in only a very weak position to do). The construction $g_{ij}x^ix^j$ transforms by invariance provided we assume

- the x^i transform as components of a weightless contravariant vector;
- the g_{ij} transform as... a weightless covariant tensor.

We are then forced by (71) to assume that

- the $\boldsymbol{e}_i$ transform as... a weightless covariant vector.

Looking to Clifford's construction

$$\boldsymbol{c} \equiv s\,\mathbf{I} + v^m\boldsymbol{e}_m + p\boldsymbol{f}$$

the established invariance of $v^m\boldsymbol{e}_m$ makes it natural to assume that

- s and $\mathbf{I}$ transform by invariance.

And since, as remarked above,

- $\boldsymbol{f}$ transforms as a scalar *density of weight* $W = +1$

we are forced to stipulate that

- p transforms as a scalar *density of weight* $W = -1$.

The point of my s, v, p-notation is now clear: those symbols are intended to suggest *scalar, vector* and *pseudo-scalar*, respectively.

Let the *conjugate* of $\boldsymbol{c}$ be defined/denoted

$$\bar{\boldsymbol{c}} \equiv s\,\mathbf{I} - v^m \boldsymbol{e}_m - p\,\boldsymbol{f}$$

From (75)—which we are now in position to write

$$\begin{aligned} \boldsymbol{c}\boldsymbol{C} = &\big[sS + v^n V_n - gpP\big]\,\mathbf{I} \\ &+ \big[sV^n + v^n S + \varepsilon^{mn} v_m P - p\varepsilon^{mn} V_m\big]\,\boldsymbol{e}_n \\ &+ \big[sP + g^{-1}\varepsilon^{mn} v_m V_n + pS\big]\,\boldsymbol{f} \end{aligned} \tag{75.2}$$

—it then follows that

$$\begin{aligned} \boldsymbol{c}\bar{\boldsymbol{c}} &= \big[s^2 - (v,v) + gp^2\big]\,\mathbf{I} = \bar{\boldsymbol{c}}\boldsymbol{c} \\ &= N(\boldsymbol{c})\cdot\mathbf{I} \end{aligned} \tag{76}$$

where

$$N(\boldsymbol{c}) \equiv s^2 - (v,v) + gp^2 \tag{77}$$

defines the *norm* of $\boldsymbol{c}$. From (76) we see that $\mathcal{C}_2[\boldsymbol{g}]$—the Clifford algebra of order 2 with arbitrary metric $\boldsymbol{g}$—is <u>not a division algebra</u> (but becomes one when the v^i are made imaginary:[13] not just the zero element, but) all elements with

$$(v,v) = s^2 + gp^2 \quad : \quad \text{defines a quadratic surface in } v\text{-space}$$

are non-invertible.

Let

$$\begin{aligned} N(\boldsymbol{c} - \lambda\mathbf{I}) &= \lambda^2 - 2s\lambda + [s^2 - (v,v) + gp^2] \\ &= \lambda^2 - 2\,\mathrm{tr}(\boldsymbol{c})\cdot\lambda + N(\boldsymbol{c}) \end{aligned} \tag{78}$$

define the "characteristic polynomial" of the Clifford number $\boldsymbol{c}$. A quick calculation serves to establish that

$$\boldsymbol{c}^2 - 2\,\mathrm{tr}(\boldsymbol{c})\cdot\boldsymbol{c} + N(\boldsymbol{c})\,\mathbf{I} = \mathbf{0} \quad : \quad \text{all Clifford numbers } \boldsymbol{c} \tag{79}$$

In short: *Every Clifford number satisfies its own characteristic equation.*[14] The zeros of (78) lie at

$$\lambda = s \pm \sqrt{(v,v) - gp^2} \tag{80}$$

[13] One is brought thus back to $\mathcal{Q}$ in the Euclidean case.

[14] My guess is that it was as the quaternionic instance of this statement—not as a proposition about matrices (that was Cayley's contribution)—that Hamilton knew the "Cayley-Hamilton theorem."

Certainly we *expect* to have

$$N(\boldsymbol{cC}) = N(\boldsymbol{c})\cdot N(\boldsymbol{C}) \tag{81}$$

but the direct demonstration (I know of no cunningly indirect demonstration) is a bit tedious. We have

$$\begin{aligned}
&N(\boldsymbol{cC}) - N(\boldsymbol{c})\cdot N(\boldsymbol{C})\\
&\quad = \Big\{ \big[s^2S^2 + 2sS(v,V) - 2gsSpP + (v,V)^2 - 2gpP(v,V) + g^2p^2P^2\big]\\
&\qquad - \big[s^2(V,V) + 2sS(v,V) + 2sP(v_m\varepsilon^{mn}V_n) - 0\\
&\qquad\qquad + S^2(v,v) + 0 + 2pS(v_m\varepsilon^{mn}V_n)\\
&\qquad\qquad + P^2(\varepsilon_k{}^m\varepsilon^{kn}v_mv_n) - 2pP(\varepsilon_k{}^m\varepsilon^{kn}v_mV_n)\\
&\qquad\qquad + p^2(\varepsilon_k{}^m\varepsilon^{kn}V_mV_n)\big]\\
&\qquad + g\big[s^2P^2 + 2g^{-1}sP(v_m\varepsilon^{mn}V_m) + 2spSP\\
&\qquad\qquad + g^{-2}(v_m\varepsilon^{mn}V_n)^2 + 2g^{-1}pS(v_m\varepsilon^{mn}V_n) + p^2S^2\big]\Big\}\\
&\qquad\qquad - \Big\{s^2S^2 - s^2(V,V) + gs^2P^2\\
&\qquad\qquad\qquad - S^2(v,v) + (v,v)(V,V) - gP^2(v,v)\\
&\qquad\qquad\qquad + gp^2S^2 - gp^2(V,V) + g^2p^2P^2\Big\}
\end{aligned}$$

which after much cancellation becomes

$$\begin{aligned}
&= \big[g^{-1}(v_m\varepsilon^{mn}V_n)^2 + (v,V)^2 - (v,v)(V,V)\big]\\
&\qquad + 2gpP\big[g^{-1}(\varepsilon_k{}^m\varepsilon^{kn}v_mV_n) - (v,V)\big]\\
&\qquad - gP^2\big[g^{-1}(\varepsilon_k{}^m\varepsilon^{kn}v_mv_n) - (v,v)\big]\\
&\qquad - gp^2\big[g^{-1}(\varepsilon_k{}^m\varepsilon^{kn}V_mV_n) - (V,V)\big]\\
&= \big[g^{-1}\varepsilon^{mn}\varepsilon^{ij} + g^{mj}g^{ni} - g^{mi}g^{nj}\big]v_mv_iV_nV_j\\
&\qquad + 2gpP\big[g^{-1}\varepsilon_k{}^m\varepsilon^{kn} - g^{mn}\big]v_mV_n\\
&\qquad - gP^2\big[g^{-1}\varepsilon_k{}^m\varepsilon^{kn} - g^{mn}\big]v_mv_n\\
&\qquad - gp^2\big[g^{-1}\varepsilon_k{}^m\varepsilon^{kn} - g^{mn}\big]V_mV_n
\end{aligned} \tag{82}$$

It becomes clear on a moment's thought that—in 2-dimensional instance of a very general proposition (see equation (21) in the material cited on page 19)—

$$\varepsilon^{i_1i_2}\epsilon_{j_1j_2} = g^{-1}\varepsilon^{i_1i_2}\varepsilon_{j_1j_2} = \begin{vmatrix}\delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2}\\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2}\end{vmatrix} = \delta^{i_1}{}_{j_1}\delta^{i_2}{}_{j_2} - \delta^{i_1}{}_{j_2}\delta^{i_2}{}_{j_1}$$

and therefore that

$$\begin{aligned}
g^{-1}\varepsilon^{mn}\varepsilon^{ij} &= g^{mi}g^{nj} - g^{mj}g^{ni}\\
g^{-1}\varepsilon_k{}^m\varepsilon^{kn} &= \delta_k{}^kg^{mn} - \delta_k{}^ng^{mk} = (2-1)g^{mn} = g^{mn}
\end{aligned}$$

Returning with this information to (82) we find that that all the [stuff]-terms vanish, completing the proof of (81).[15]

Proceeding on the assumption that $\boldsymbol{u}$ is an *invertible* Clifford number,[16] we look now again to transformations of the form

$$\boldsymbol{c} \longmapsto \boldsymbol{C} = \boldsymbol{u}^{-1}\boldsymbol{c}\,\boldsymbol{u} \tag{83}$$

Such transformations are, by (81), *norm-preserving*. It is clear also that we can, without loss of generality, assume $\boldsymbol{u}$ to be unimodular: $N(\boldsymbol{u}) = 1$. Equation (83) sets up a linear relationship between the elements of $\boldsymbol{C}$ and those of $\boldsymbol{c}$, which we emphasize by writing

$$\begin{pmatrix} s \\ v^1 \\ v^2 \\ p \end{pmatrix} \longmapsto \begin{pmatrix} S \\ V^1 \\ V^2 \\ P \end{pmatrix} = \mathbb{U}\begin{pmatrix} s \\ v^1 \\ v^2 \\ p \end{pmatrix} \tag{84}$$

The norm of $\boldsymbol{c}$ can in this notation be written

$$N(\boldsymbol{c}) = \begin{pmatrix} s \\ v^1 \\ v^2 \\ p \end{pmatrix}^{\!\mathsf{T}} \mathbb{G} \begin{pmatrix} s \\ v^1 \\ v^2 \\ p \end{pmatrix} \quad \text{with} \quad \mathbb{G} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -g_{11} & -g_{12} & 0 \\ 0 & -g_{21} & -g_{22} & 0 \\ 0 & 0 & 0 & g \end{pmatrix} \tag{85}$$

(notice that the real symmetric matrix $\mathbb{G}$ gives back (36) in the Euclidean case) and the representation (84) of (83) will itself be norm-preserving if and only if $\mathbb{U}$ is $\mathbb{G}$-orthogonal: $\mathbb{U}^{\mathsf{T}}\mathbb{G}\mathbb{U} = \mathbb{G}$. This was seen already on page to require that the infinitesimal generator of $\mathbb{U}$ be $\mathbb{G}$-antisymmetric. It is with those points fresh in our minds that we turn to the details.

Write

$$\boldsymbol{u} = \mathsf{I} + \epsilon\,\boldsymbol{w} \quad : \quad \text{neglect terms of order } \epsilon^2 \tag{86}$$

where *epsilon* is an infinitesmal parameter (that, since it wears no indices, will not be confused with the Levi-Civita tensor). In leading order $\boldsymbol{u}^{-1} = \mathsf{I} - \epsilon\,\boldsymbol{w}$ and (83) becomes

$$\boldsymbol{c} \longmapsto \boldsymbol{C} = \boldsymbol{c} + \epsilon[\boldsymbol{c}\,\boldsymbol{w} - \boldsymbol{w}\,\boldsymbol{c}] + \cdots \tag{87}$$

Write (to establish our notational conventions)

$$\begin{aligned} \boldsymbol{c} &= s\,\mathsf{I} + v^n\boldsymbol{e}_n + p\boldsymbol{f} \\ \boldsymbol{w} &= \sigma\,\mathsf{I} + w^n\boldsymbol{e}_n + w\boldsymbol{f} \end{aligned}$$

---

[15] The argument presented above tells us nothing useful about *why* (81) is valid, and can be expected to become rapidly more difficult to carry to completion as the order of the Clifford algebra ascends. What we need—but what I presently lack—is a *simple, illuminating, dimensionally generalizable* proof of fundamental statement (81).

[16] Note that the set of invertible Clifford numbers has *group* structure with respect to the operation of multiplication. I'm sure mathematicians must have a name for such things.

and from (75.2) obtain

$$[\boldsymbol{c}\,\boldsymbol{w} - \boldsymbol{w}\,\boldsymbol{c}] = 2\big[w\varepsilon^{mn}v_m - w_m\varepsilon^{mn}p\big]\boldsymbol{e}_n - 2g^{-1}\big[w_m\varepsilon^{mn}v_n\big]\boldsymbol{f} \tag{88.1}$$

$$= 2g\big[w\epsilon_{mn}v^m - w^m\epsilon_{mn}p\big]g^{nk}\boldsymbol{e}_k - 2\big[w^m\epsilon_{mn}v^n\big]\boldsymbol{f} \tag{88.2}$$

Notice that σ is silent: it is without loss of generality that we henceforth assume $\boldsymbol{w}$ to be spurless, writing

$$\boldsymbol{w} = w^n\boldsymbol{e}_n + w\boldsymbol{f} \tag{89.1}$$

where, by the assumed unimodularity of $\boldsymbol{w}$,

$$(w)^2 = 1 + w^m g_{mn} w^n \tag{89.2}$$

Notice also that—relatedly—s does not participate in the transformation (87); *i.e.*, that it transforms by invariance (which is to say: "like a scalar"). Notice finally that because

- g has weight $W = 2$
- ε^{mn} and $\boldsymbol{f}$ have weight $W = 1$
- v^n, w^n, g^{mn} and $\boldsymbol{e}_n$ are weightless: $W = 0$
- ϵ_{mn}, p and w have weight $W = -1$
- g^{-1} has weight $W = -2$

each of the terms on the right side of (88) is—as we require—weightless.

In representation of (86) we have

$$\mathbb{U} = \mathbb{I} + 2\epsilon\mathbb{W} \tag{90.1}$$

and write

$$\mathbb{W} = w^n\mathbb{J}_n + w\mathbb{K} \tag{90.2}$$

to emphasize the fact that $\mathbb{W}$ depends *linearly* on the coordinates of $\boldsymbol{w}$. The detailed designs of $\mathbb{J}_1$, $\mathbb{J}_2$ and $\mathbb{K}$ can be read off from (88), which supplies

$$\left.\begin{aligned}
\mathbb{J}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g\cdot g^{21} \\ 0 & 0 & 0 & -g\cdot g^{22} \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
\mathbb{J}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +g\cdot g^{11} \\ 0 & 0 & 0 & +g\cdot g^{12} \\ 0 & +1 & 0 & 0 \end{pmatrix} \\
\mathbb{K} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & g\cdot g^{21} & -g\cdot g^{11} & 0 \\ 0 & g\cdot g^{22} & -g\cdot g^{12} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}\right\} \tag{91}$$

Quick (but interesting) calculation confirms that each of the matrices (91) is in fact $\mathbb{G}$-antisymmetyric, and they are seen to give back the matrices encountered on page 6 in the Euclidean case. *Mathematica*-assisted calculation gives

$$\left.\begin{aligned} \mathbb{J}_1\mathbb{J}_2 - \mathbb{J}_2\mathbb{J}_1 &= -\mathbb{K} \\ \mathbb{J}_2\mathbb{K} - \mathbb{K}\,\mathbb{J}_2 &= g(g^{11}\mathbb{J}_1 + g^{12}\mathbb{J}_2) \\ \mathbb{K}\,\mathbb{J}_1 - \mathbb{J}_1\mathbb{K} &= g(g^{21}\mathbb{J}_1 + g^{22}\mathbb{J}_2) \end{aligned}\right\} \tag{92}$$

which assume the simple form (39) in the Euclidean case. Also

$$\begin{aligned} \det(\mathbb{J}_1 - \lambda\mathbb{I}) &= \lambda^4 - g{\cdot}g^{22}\lambda^2 \\ \det(\mathbb{J}_2 - \lambda\mathbb{I}) &= \lambda^4 - g{\cdot}g^{11}\lambda^2 \\ \det(\mathbb{K} - \lambda\mathbb{I}) &= \lambda^4 + g\lambda^2 \end{aligned}$$

which in the Euclidean case reproduce results reported on page 7. Finally we have

$$\begin{aligned} \det(s\,\mathbb{I} + v^1\mathbb{J}_1 + v^2\mathbb{J}_2 + p\,\mathbb{K}) &= s^2(s^2 - v^m g_{mn} v^m + gp^2) \\ &= s^2\cdot N(s\,\mathbf{I} + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\,\boldsymbol{f}) \end{aligned} \tag{93}$$

by simplification[17] of the result reported by *Mathematica*.

We are brought thus to the conclusion that

$$\boldsymbol{c} \longmapsto \boldsymbol{C} = e^{-\theta\boldsymbol{w}}\boldsymbol{c}\,e^{\theta\boldsymbol{w}} \quad : \quad \boldsymbol{w} = w^n\boldsymbol{e}_n + w\boldsymbol{f}$$

and

$$\begin{pmatrix} s \\ v^1 \\ v^2 \\ p \end{pmatrix} \longmapsto \begin{pmatrix} S \\ V^1 \\ V^2 \\ P \end{pmatrix} = e^{2\theta\mathbb{W}} \begin{pmatrix} s \\ v^1 \\ v^2 \\ p \end{pmatrix} \quad : \quad \mathbb{W} = w^n\mathbb{J}_n + w\,\mathbb{K}$$

say the same thing in two different ways.

Such transformations generally involve "vector/psuedoscalar intermixing," and in that respect relate unnaturally to our point of departure, which was the $\mathbb{g}$-rotationally invariant expression $x^m g_{mn} x^n$. Transformations "natural" to that expression result in the present formalism from setting $v^1 = v^2 = 0$ and $w = 1$, in which connection I note that *Mathematica*'s `MatrixExp` command instantly produces

$$e^{2\theta\mathbb{K}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \text{stuff} & \text{stuff} & 0 \\ 0 & \text{stuff} & \text{stuff} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the unsimplified "stuff" terms are enormously complicated. We have

---

[17] Use

$$g\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$$

touched here upon a subject of some intrinsic interest, so I linger to develop some of the details:

We need only concern ourselves with the 2×2 "nucleus" of $\mathbb{K}$; *i.e.*, with

$$\mathbb{k} \equiv g \begin{pmatrix} g^{21} & -g^{11} \\ g^{22} & -g^{12} \end{pmatrix} = \begin{pmatrix} -g_{21} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix}$$

which is readily seen to be $\mathcal{g}$-antisymmetric:

$$\mathcal{g}\mathbb{k} = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix} \text{ is antisymmetric}$$

From $\det(\mathbb{k} - \lambda\mathbb{I}) = \lambda^2 + g$ it follows that $\mathbb{k}^2 + g\,\mathbb{I} = \mathbb{O}$. It is natural, therefore, to introduce $\hat{\mathbb{k}} \equiv \frac{1}{\sqrt{g}}\,\mathbb{k}$ since it satisfies the simpler equation $\hat{\mathbb{k}}^2 + \mathbb{I} = \mathbb{O}$. We now have

$$\begin{aligned} \mathcal{u} &\equiv \exp\big\{2\theta\,\mathbb{k}\big\} \\ &= \exp\big\{2\vartheta\,\hat{\mathbb{k}}\big\} \quad \text{with} \quad \vartheta \equiv \sqrt{g}\,\theta \\ &= \cos 2\vartheta \cdot \mathbb{I} + \sin 2\vartheta \cdot \hat{\mathbb{k}} \\ &= \cos 2\vartheta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin 2\vartheta}{\sqrt{g}} \begin{pmatrix} -g_{21} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix} \end{aligned} \tag{94}$$

and verify that $\mathcal{u}^{\mathsf{T}}\mathcal{g}\,\mathcal{u} = \mathcal{g}$. In the Euclidean case

$$\mathcal{g} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad : \quad \sqrt{g} = 1$$

we recover the rotation matrix

$$\mathcal{u} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

while specialization to the Minkowski metric

$$\mathcal{g} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad : \quad \sqrt{g} = i$$

gives the Lorentz matrix

$$\mathcal{u} = \begin{pmatrix} \cos 2i\theta & i^{-1}\sin 2i\theta \\ i^{-1}\sin 2i\theta & \cos 2i\theta \end{pmatrix} = \begin{pmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{pmatrix}$$

If we were Dirac-like inhabitants of a 1-dimensional world (2-dimensional spacetime) we would have essential interest in the least-dimensional *matrix representations of the fundamental anticommutation relations* (72). I turn now to description of a method for construcing such matrices.[18] We proceed from the observation that these real, symmetric, traceless matrices

$$\mathbb{e}_1'' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{e}_2'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{95.1}$$

—convenient variants of some $\mathbb{e}$-matrices introduced at (45)—would serve our needs in the Euclidean case

$$\mathbb{g}_{\text{Euclidean}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for by calculation

$$\begin{aligned} \mathbb{e}_1''\mathbb{e}_1'' &= \quad g_{11} \cdot \mathbb{I} = 1 \cdot \mathbb{I} \\ \mathbb{e}_2''\mathbb{e}_2'' &= \quad g_{22} \cdot \mathbb{I} = 1 \cdot \mathbb{I} \\ \mathbb{e}_1''\mathbb{e}_2'' + \mathbb{e}_2''\mathbb{e}_1'' &= 2\, g_{12} \cdot \mathbb{I} = 0 \cdot \mathbb{I} \end{aligned}$$

Therefore the matrices

$$\mathbb{e}_1' \equiv \sqrt{g_1}\, \mathbb{e}_1'' \quad \text{and} \quad \mathbb{e}_2' \equiv \sqrt{g_2}\, \mathbb{e}_2'' \tag{95.2}$$

serve our needs in the diagonal case

$$\mathbb{g}_{\text{diagonal}} = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

for trivially

$$\begin{aligned} \mathbb{e}_1'\mathbb{e}_1' &= \quad g_{11} \cdot \mathbb{I} = g_1 \cdot \mathbb{I} \\ \mathbb{e}_2'\mathbb{e}_2' &= \quad g_{22} \cdot \mathbb{I} = g_2 \cdot \mathbb{I} \\ \mathbb{e}_1'\mathbb{e}_2' + \mathbb{e}_2'\mathbb{e}_1' &= 2\, g_{12} \cdot \mathbb{I} = \ 0 \ \cdot \mathbb{I} \end{aligned}$$

Now construct

$$\left.\begin{aligned} \mathbb{e}_1 &\equiv \mathbb{e}_1' \cos\alpha - \mathbb{e}_2' \sin\alpha \\ \mathbb{e}_2 &\equiv \mathbb{e}_1' \sin\alpha + \mathbb{e}_2' \cos\alpha \end{aligned}\right\} \tag{95.3}$$

and from the requirements

$$\begin{aligned} \mathbb{e}_1\mathbb{e}_1 &= \quad g_{11} \cdot \mathbb{I} \\ \mathbb{e}_1\mathbb{e}_2 + \mathbb{e}_2\mathbb{e}_1 &= 2\, g_{12} \cdot \mathbb{I} \\ \mathbb{e}_2\mathbb{e}_2 &= \quad g_{22} \cdot \mathbb{I} \end{aligned}$$

obtain

$$\left.\begin{aligned} g_{11} &= g_1 \cos^2\alpha + g_2 \sin^2\alpha = \tfrac{1}{2}(g_1 + g_2) + \tfrac{1}{2}(g_1 - g_2)\cos 2\alpha \\ g_{12} = g_{21} &= (g_1 - g_2)\cos\alpha \sin\alpha \ = \tfrac{1}{2}(g_1 - g_2)\sin 2\alpha \\ g_{22} &= g_1 \sin^2\alpha + g_2 \cos^2\alpha = \tfrac{1}{2}(g_1 + g_2) - \tfrac{1}{2}(g_1 - g_2)\cos 2\alpha \end{aligned}\right\} \tag{96}$$

[18] My primary source will be some penciled notes I wrote in October, 1994, in response to points raised in David Griffiths' elementary particles course, which I attended that term.

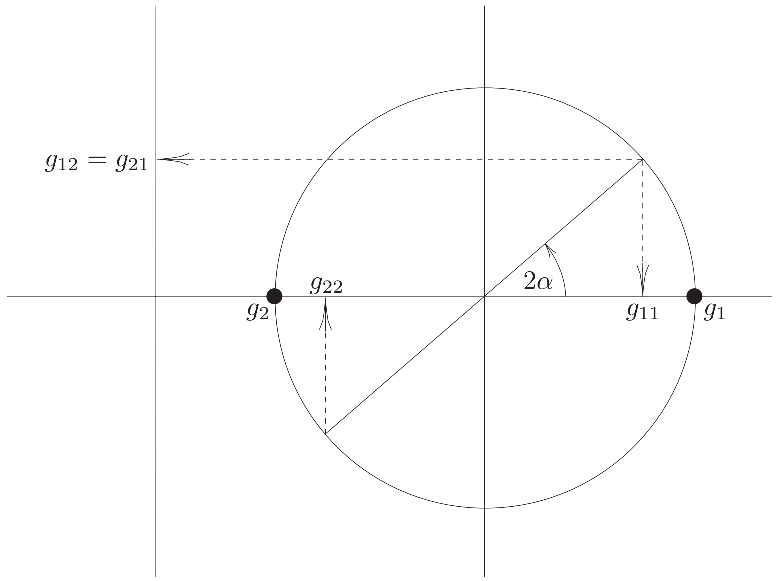

FIGURE 1: *Diagrammatic interpretation of* (96)*, known to engineers as* **Mohr's construction**. *The circle is centered at* $\frac{1}{2}(g_1+g_2)$ *and has radius* $\frac{1}{2}(g_1-g_2)$. *Reading* g_{11}, g_{22} *and* $g_{12}=g_{21}$ *from the figure, one obtains precisely* (96).

Evidently the triples $\{g_{11},\, g_{12}=g_{21},\, g_{22}\}$ and $\{g_1,\, g_2,\, \alpha\}$ provide *alternative but equivalent descriptions* of the 2×2 real symmetric matrix $\mathbb{g}$. This fact has been known and used for well more than a century by engineers, and its elegant diagrammatic interpretation—see the figure—is known as "Mohr's construction."[19] *Mathematica* confirms, by the way, that matrices of the design

$$\begin{pmatrix} g_1\cos^2\alpha+g_2\sin^2\alpha & (g_1-g_2)\cos\alpha\sin\alpha \\ (g_1-g_2)\cos\alpha\sin\alpha & g_1\sin^2\alpha+g_2\cos^2\alpha \end{pmatrix}$$

have eigenvalues $\{g_1,\, g_2\}$ for all values of α. The conclusion of interest is that

$$\left.\begin{aligned} \mathbb{e}_1 &\equiv \begin{pmatrix} \sqrt{g_1}\cos\alpha & -\sqrt{g_2}\,\sin\alpha \\ -\sqrt{g_2}\,\sin\alpha & -\sqrt{g_1}\,\cos\alpha \end{pmatrix} \\ \mathbb{e}_2 &\equiv \begin{pmatrix} \sqrt{g_1}\,\sin\alpha & \sqrt{g_2}\cos\alpha \\ \sqrt{g_2}\cos\alpha & -\sqrt{g_1}\,\sin\alpha \end{pmatrix} \end{aligned}\right\} \tag{97}$$

[19] See "Non-standard applications of Mohr's construction" (1998). Mohr was a professor of civil engineering first in Stuttgart, later in Dresden, and was led to his construction (1882) as a means of clarifying a problem having to do with the fracture of brittle materials. In some respects he had been anticipated by another civil engineer named Culmann (1866). Both were studying a problem that had been pioneered by Coulomb.

Which, indeed, check out: working from (97) we find

$$\boldsymbol{e}_1\boldsymbol{e}_1 = \begin{pmatrix} g_1\cos^2\alpha + g_2\sin^2\alpha & 0 \\ 0 & g_1\cos^2\alpha + g_2\sin^2\alpha \end{pmatrix} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{11} \end{pmatrix}$$

$$\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1 = \begin{pmatrix} (g_1 - g_2)\sin 2\alpha & 0 \\ 0 & (g_1 - g_2)\sin 2\alpha \end{pmatrix} = 2\begin{pmatrix} g_{12} & 0 \\ 0 & g_{12} \end{pmatrix}$$

$$\boldsymbol{e}_2\boldsymbol{e}_2 = \begin{pmatrix} g_1\sin^2\alpha + g_2\cos^2\alpha & 0 \\ 0 & g_1\sin^2\alpha + g_2\cos^2\alpha \end{pmatrix} = \begin{pmatrix} g_{22} & 0 \\ 0 & g_{22} \end{pmatrix}$$

Our 1-dimensional Dirac would set $g_1 = +1$, $g_2 = -1$, $\alpha = 0$ and by (97) obtain

$$\mathit{\Gamma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\mathit{\Gamma}_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which again check out:

$$\mathit{\Gamma}_1\mathit{\Gamma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathit{\Gamma}_1\mathit{\Gamma}_2 + \mathit{\Gamma}_2\mathit{\Gamma}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\mathit{\Gamma}_2\mathit{\Gamma}_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the notes cited previously[18] I work out in fair detail the theory of the resulting "Dirac equation"

$$(\mathit{\Gamma}^m\partial_m + i\varkappa\mathbb{I})\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{98}$$

There are no major surprises. Our 1-dimensional physicists might, however, be surprised by their discovery that transformations that are $\mathbb{G}$-orthogonal with respect to (see again (85)) the "hyperdimensional metric"

$$\mathbb{G}_{\text{dirac}} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

are latent in the design of their little theory.

Here—though one aspect of the subject remains to be developed—I bring to an end this review of the $\mathcal{C}_2[\boldsymbol{g}]$ generated by $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. That this discussion, which began on page 17, progressed as smoothly as it did can, I think, be attributed mainly to the fact that at (74) we chose $\boldsymbol{f} = \boldsymbol{e}_1\boldsymbol{e}_2 - g_{12}\mathbf{I}$ (rather than $\boldsymbol{e}_1\boldsymbol{e}_2$) to close the algebra.

5. Third order Clifford algebra with general metric. Equations (71) and (72)—

$$(g_{ij}x^i x^j) = (x^i \boldsymbol{e}_i)^2 \quad \text{whence} \quad \boldsymbol{e}_i\,\boldsymbol{e}_j + \boldsymbol{e}_j\,\boldsymbol{e}_i = 2g_{ij}\,\mathbf{I}$$

—remain in force, the difference being that all indices range now on $\{1, 2, 3\}$. $\mathcal{C}_3[\boldsymbol{g}]$ is an algebra of order $2^3 = 8$, generated by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. The initial question is: *How most usefully to describe the general element* $\boldsymbol{c}$ *of* $\mathcal{C}_3[\boldsymbol{g}]$? How most naturally to achieve algebraic *closure*? I propose to adopt a practice standard to the exterior calculus. Let

$$\begin{aligned} \boldsymbol{e}_{i_1 i_2 \ldots i_p} &\equiv \tfrac{1}{p!}\,\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_p} \\ &\equiv \tfrac{1}{p!}\Big\{\text{antisymmetrized} \sum_{\text{permutations}}\Big\} \end{aligned} \tag{99}$$

which carries with it the implication that in the n-dimensional case there will be $\binom{n}{p}$ distinct terms of order p. Look back again to the

Case $n = 2$ We have one element $\mathbf{I}$ of order $p = 0$, two elements $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ of order $p = 1$, and one element

$$\begin{aligned} \boldsymbol{e}_{12} &= \tfrac{1}{2!}(\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1) \\ &= \tfrac{1}{2!}(2\,\boldsymbol{e}_1\boldsymbol{e}_2 - 2g_{12}\,\mathbf{I}) = \boldsymbol{f} \end{aligned}$$

of order $p = 2$ (which, however, has two distinct names: $\boldsymbol{e}_{12} = -\boldsymbol{e}_{21}$). Proceeding similarly to the case of immediate interest, we in

Case $n = 3$ have one element $\mathbf{I}$ of order $p = 0$, three elements $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of order $p = 1$, three elements

$$\begin{aligned} \boldsymbol{e}_{23} &= \tfrac{1}{2!}(\boldsymbol{e}_2\boldsymbol{e}_3 - \boldsymbol{e}_3\boldsymbol{e}_2) \\ &= \tfrac{2}{2!}(\,\boldsymbol{e}_2\boldsymbol{e}_3 - g_{23}\,\mathbf{I}) = -\boldsymbol{e}_{32} \\ \boldsymbol{e}_{13} &= \tfrac{1}{2!}(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_3\boldsymbol{e}_1) \\ &= \tfrac{2}{2!}(\,\boldsymbol{e}_1\boldsymbol{e}_3 - g_{13}\,\mathbf{I}) = -\boldsymbol{e}_{31} \\ \boldsymbol{e}_{12} &= \tfrac{1}{2!}(\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1) \\ &= \tfrac{2}{2!}(\,\boldsymbol{e}_1\boldsymbol{e}_2 - g_{12}\,\mathbf{I}) = -\boldsymbol{e}_{21} \end{aligned}$$

of order $p = 2$, and one element

$$\begin{aligned} \boldsymbol{e}_{123} &= \tfrac{1}{3!}(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_3\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1) \\ &= \tfrac{6}{3!}(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - g_{23}\boldsymbol{e}_1 + g_{31}\boldsymbol{e}_2 - g_{12}\boldsymbol{e}_3) \end{aligned}$$

of order $p = 3$ (which has 3! different names). It becomes natural in this light to write

$$\boldsymbol{c} = c\,\mathbf{I} + c^i\boldsymbol{e}_i + \tfrac{1}{2!}c^{ij}\boldsymbol{e}_{ij} + \tfrac{1}{3!}c^{ijk}\boldsymbol{e}_{ijk} \tag{100}$$

where the coefficients are *weightless antisymmetric tensors*[20] of ascending order and we have politely "averaged over all alternative names."

But adoption of such a policy would carry with it the implication that to describe the elements of $\mathcal{C}_2[\boldsymbol{g}]$ we should write

$$\boldsymbol{c} = c\,\mathbf{I} + c^i\boldsymbol{e}_i + \tfrac{1}{2!}c^{ij}\boldsymbol{e}_{ij}$$

whereas it has been our established practice to write $\boldsymbol{c} \equiv s\,\mathbf{I} + v^i\boldsymbol{e}_i + p\boldsymbol{f}$. How to achieve consistency? Let

$$\begin{aligned}\boldsymbol{f}^{i_1i_2\ldots i_{n-p}} &\equiv g^{-\frac{1}{2}}\cdot\tfrac{1}{p!}\varepsilon^{i_1i_2\ldots i_{n-p}j_1j_2\ldots j_p}\boldsymbol{e}_{j_1j_2\ldots j_p}\\ &= g^{+\frac{1}{2}}\cdot\tfrac{1}{p!}\epsilon^{i_1i_2\ldots i_{n-p}j_1j_2\ldots j_p}\boldsymbol{e}_{j_1j_2\ldots j_p}\end{aligned} \tag{101}$$

define the population $\{\boldsymbol{f}^{i_1i_2\ldots i_{n-p}}\}$ of elements *dual* to the population $\{\boldsymbol{e}_{i_1i_2\ldots i_p}\}$. The $g^{\pm\frac{1}{2}}$-factors have been introduced to insure that elements of a population and its dual transform with the same weight (which is to say: weightlessly). The two populations contain identically many elements: $\binom{n}{n-p} = \binom{n}{p}$. But each element of $\{\boldsymbol{e}_{i_1i_2\ldots i_p}\}$ has $p!$ distinct names, while each element of the dual population $\{\boldsymbol{f}^{i_1i_2\ldots i_{n-p}}\}$ has $(n-p)!$ distinct names: that distinction is greatest at $p = n$, and it disappears at $p = \frac{1}{2}n$ (which requires that n be even). From[21]

$$\begin{aligned}g^{+\frac{1}{2}}&\tfrac{1}{(n-p)!}\epsilon_{k_1k_2\ldots k_pi_1i_2\ldots i_{n-p}}\boldsymbol{f}^{i_1i_2\ldots i_{n-p}}\\ &= g\,\tfrac{1}{p!(n-p)!}\epsilon_{k_1k_2\ldots k_pi_1i_2\ldots i_{n-p}}\epsilon^{i_1i_2\ldots i_{n-p}j_1j_2\ldots j_p}\boldsymbol{e}_{j_1j_2\ldots j_p}\\ &= \tfrac{1}{p!(n-p)!}(-)^{p(n-p)}g\epsilon_{k_1k_2\ldots k_pi_1i_2\ldots i_{n-p}}\epsilon^{j_1j_2\ldots j_pi_1i_2\ldots i_{n-p}}\boldsymbol{e}_{j_1j_2\ldots j_p}\\ &= (-)^{p(n-p)}\tfrac{1}{p!}\delta_{k_1k_2\ldots k_p}{}^{j_1j_2\ldots j_p}\boldsymbol{e}_{j_1j_2\ldots j_p}\\ &= (-)^{p(n-p)}\boldsymbol{e}_{k_1k_2\ldots k_p}\end{aligned} \tag{102}$$

we see that "double dualization" returns the original population except, perhaps, for an overall sign—a minus sign that is present if and only if n is even and p is odd.

Look in particular to the case that precipitated this discussion: the case $n = p = 2$. Drawing upon (101) and (102) we have

$$\begin{aligned}\boldsymbol{f} &= g^{-\frac{1}{2}}\cdot\tfrac{1}{2}\varepsilon^{j_1j_2}\,\boldsymbol{e}_{j_1j_2} = g^{-\frac{1}{2}}\cdot\tfrac{1}{2}(\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1)\\ &= g^{+\frac{1}{2}}\cdot\tfrac{1}{2}\epsilon^{j_1j_2}\,\boldsymbol{e}_{j_1j_2}\end{aligned}$$

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[20] Use of the term "tensor" will remain technically unwarrented until we have given explicit attention to the *transformational* aspects of the theory.

[21] Here I allow myself to make free use of notions (for example: that of the "generalized Kronecker delta") and identities—workhorses of exterior algebra—that (as was mentioned already on page 19) are developed on pages 7–9 of "Electrodynamical applications of the exterior calculus" (1996).

(note that the first of those equations differs from (74) only by the inclusion of the weight-preserving $\sqrt{g}$-factor) and

$$\boldsymbol{e}_{k_1k_2} = (-)^2 g^{+\frac{1}{2}}\cdot\epsilon_{k_1k_2}\boldsymbol{f}$$

(note that, because $\boldsymbol{e}_{k_1k_2}$ wears what is in case $n = 2$ a full complement of indices, $\boldsymbol{f}$ is deprived of *any*). Introducing this last bit of information into

$$\boldsymbol{c} = c\,\mathbf{I} + c^i\boldsymbol{e}_i + \tfrac{1}{2!}c^{ij}\boldsymbol{e}_{ij}$$

we obtain

$$= c\,\mathbf{I} + c^i\boldsymbol{e}_i + \big\{\sqrt{g}\tfrac{1}{2!}\epsilon_{ij}c^{ij}\big\}\boldsymbol{f} \tag{103.1}$$

which differs only notationally from our former

$$= s\,\mathbf{I} + v^i\boldsymbol{e}_i + p\boldsymbol{f} \tag{103.2}$$

except in this detail: $\sqrt{g}$-factors have served in (103.1) to render both $\big\{etc.\big\}$ and $\boldsymbol{f}$ weightless, while the $\boldsymbol{f}$ in (103.2) has weight $W = +1$ and its coefficient p has weight $W = -1$. We confront therefore a

> **POLICY DECISION:** Should or should not $\sqrt{g}$-factors be included? Inclusion seems to simplify discussion of general algebraic issues, but in cases where $g < 0$ serves to introduce i's that for physical reasons may be unwelcome. My policy will be to retain the $\sqrt{g}$'s, with the understanding that in specific applications we may want to drop them. The practice of writing $\sqrt{|g|}$ that is sometimes used in general relativity seems to me to create more problems than it solves.

In the past—especially when working in $\mathcal{C}_4[\boldsymbol{g}]$—I have found it most convenient to adopt the "symmetrized hybrid" notations that proceed

$$\begin{array}{lcl}
\text{in } \mathcal{C}_2[\boldsymbol{g}] & : & \boldsymbol{c} = s\,\mathbf{I} + s^i\boldsymbol{e}_i + p\boldsymbol{f}\\
\text{in } \mathcal{C}_3[\boldsymbol{g}] & : & \boldsymbol{c} = s\,\mathbf{I} + s^i\boldsymbol{e}_i + p^i\boldsymbol{f}_i + p\boldsymbol{f}\\
\text{in } \mathcal{C}_4[\boldsymbol{g}] & : & \boldsymbol{c} = s\,\mathbf{I} + s^i\boldsymbol{e}_i + \tfrac{1}{2!}s^{ij}\boldsymbol{e}_{ij} + p^i\boldsymbol{f}_i + p\boldsymbol{f}\\
\text{in } \mathcal{C}_5[\boldsymbol{g}] & : & \boldsymbol{c} = s\,\mathbf{I} + s^i\boldsymbol{e}_i + \tfrac{1}{2!}s^{ij}\boldsymbol{e}_{ij} + \tfrac{1}{2!}p^{ij}\boldsymbol{f}_{ij} + p^i\boldsymbol{f}_i + p\boldsymbol{f}\\
\text{in } \mathcal{C}_6[\boldsymbol{g}] & : & \boldsymbol{c} = s\,\mathbf{I} + s^i\boldsymbol{e}_i + \tfrac{1}{2!}s^{ij}\boldsymbol{e}_{ij} + \tfrac{1}{3!}s^{ijk}\boldsymbol{e}_{ijk} + \tfrac{1}{2!}p^{ij}\boldsymbol{f}_{ij} + p^i\boldsymbol{f}_i + p\boldsymbol{f}\\
 & \vdots &
\end{array}$$

These have at least the merit that they total minimize the number of indices and mimic the symmetry of the binomial distribution. The scheme does, however, become ambiguous "at the middle" when n is even (should one write $\frac{1}{3!}s^{ijk}\boldsymbol{e}_{ijk}$ or $\frac{1}{3!}p^{ijk}\boldsymbol{f}_{ijk}$?) and in some applications it presents also other disadvantages ...as will emerge. When $\boldsymbol{c}$ is presented as described above I will say it has been presented in "symmetrized form," and will use "canonical form" to refer to the presentation

$$\boldsymbol{c} = c\,\mathbf{I} + c^i\boldsymbol{e}_i + \tfrac{1}{2!}c^{ij}\boldsymbol{e}_{ij} + \tfrac{1}{3!}c^{ijk}\boldsymbol{e}_{ijk} + \tfrac{1}{4!}c^{ijkl}\boldsymbol{e}_{ijkl} + \cdots$$

Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$—elements of $\mathcal{C}_n[\mathbb{g}]$—have been presented in canonical form, and that it is desired to obtain the canonical description of their product $\boldsymbol{ab}$. What we then need (and cannot do without!) are formulæ of the type

$$\begin{aligned}\boldsymbol{e}_{i_1\ldots i_p}\cdot\boldsymbol{e}_{j_1\ldots j_q} = c\,\mathbf{I} + c^k\boldsymbol{e}_k + \tfrac{1}{2!}c^{k_1k_2}\boldsymbol{e}_{k_1k_2}\\ + \cdots + \tfrac{1}{(p+q)!}c^{k_1\ldots k_{p+q}}\boldsymbol{e}_{k_1\ldots k_{p+q}}\end{aligned} \tag{104}$$

If our interest shifted to a Clifford algebra of *higher* order then we would need those *same* formulæ *plus some of their higher order companions*, whereas if we shifted our interest to a Clifford algebra of lower order we would find that we had already in hand all the material we need...though some terms would automatically blink off because

$$\boldsymbol{e}_{k_1\ldots k_{p+q}} = \mathbf{0} \text{ if any index is repeated}$$

and if the indices range on a reduced set such repeats become unavoidable. An element of "universality" (n-independence) attaches therefore to formulæ of type (104). Note that the "symmetrized" notation does *not* lend itself well to the problem in hand, for the onset of $\boldsymbol{f}$ term *is* n-dependent. I describe an approach to the construction of such formulæ.

LOWEST LEVEL ANALYSIS We have

$$\begin{aligned}\boldsymbol{e}_1\boldsymbol{e}_2 &= \boldsymbol{e}_1\boldsymbol{e}_2\\ -\boldsymbol{e}_2\boldsymbol{e}_1 &= \boldsymbol{e}_1\boldsymbol{e}_2 - 2g_{12}\mathbf{I}\end{aligned}$$

Add and multiply by $\frac{1}{2!}$ to obtain $\boldsymbol{e}_{12} = \boldsymbol{e}_1\boldsymbol{e}_2 - g_{12}\mathbf{I}$, the general implication being that

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \boldsymbol{e}_{ij} + g_{ij}\mathbf{I} \tag{105}$$

which, by the way, follows directly from $\boldsymbol{e}_i\boldsymbol{e}_i = \frac{1}{2}(\boldsymbol{e}_i\boldsymbol{e}_j - \boldsymbol{e}_j\boldsymbol{e}_i) + \frac{1}{2}(\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i)$ and works even when $i = j$. As a check on the accuracy of (105.1) we have

$$\begin{aligned}\tfrac{1}{2!}\sum_{\text{signed permutations}}\boldsymbol{e}_i\cdot\boldsymbol{e}_j &= \tfrac{1}{2!}(\boldsymbol{e}_{ij} - \boldsymbol{e}_{ji}) + \tfrac{1}{2!}(g_{ij} - g_{ji})\,\mathbf{I}\\ &= \boldsymbol{e}_{ij} \ :\ g\text{-terms cancel by symmetry}\end{aligned}$$

NEXT HIGHER LEVEL Our objective will be to develop $\boldsymbol{e}_i\cdot\boldsymbol{e}_{jk}$ and $\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_k$. To that end, we look to each of the terms that contribute to $\boldsymbol{e}_{ijk}$ and use the "flip principle" $\boldsymbol{e}_m\boldsymbol{e}_n = -\boldsymbol{e}_n\boldsymbol{e}_m + 2g_{mn}\mathbf{I}$ to bring each $\boldsymbol{e}\cdot\boldsymbol{e}\cdot\boldsymbol{e}$ to "dictionary order." This will supply the canonical development of $\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k$, which we will use to assemble the formulæ of interest. Turning to the details, we have

$$\begin{aligned}\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\\ -\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_2 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - 2g_{23}\boldsymbol{e}_1\\ \boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 + 2g_{13}\boldsymbol{e}_2 - 2g_{12}\boldsymbol{e}_3\\ -\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_3 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - 2g_{12}\boldsymbol{e}_3\\ \boldsymbol{e}_3\boldsymbol{e}_1\boldsymbol{e}_2 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 + 2g_{13}\boldsymbol{e}_2 - 2g_{23}\boldsymbol{e}_1\\ -\boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - 2g_{23}\boldsymbol{e}_1 + 2g_{13}\boldsymbol{e}_2 - 2g_{12}\boldsymbol{e}_3\end{aligned}$$

Adding those results together and dividing by 6, we have

$$\boldsymbol{e}_{123} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - g_{12}\boldsymbol{e}_3 + g_{31}\boldsymbol{e}_2 - g_{23}\boldsymbol{e}_1$$

or

$$\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = \boldsymbol{e}_{123} + g_{12}\boldsymbol{e}_3 - g_{31}\boldsymbol{e}_2 + g_{23}\boldsymbol{e}_1$$

which in the general case reads

$$\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k = \boldsymbol{e}_{ijk} + g_{ij}\boldsymbol{e}_k - g_{ki}\boldsymbol{e}_j + g_{jk}\boldsymbol{e}_i \tag{106}$$

and gives

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_{jk} = \boldsymbol{e}_{ijk} - g_{ik}\boldsymbol{e}_j + g_{ij}\boldsymbol{e}_k \tag{107.1}$$

$$\boldsymbol{e}_{jk} \cdot \boldsymbol{e}_i = \boldsymbol{e}_{jki} + g_{ik}\boldsymbol{e}_j - g_{ij}\boldsymbol{e}_k \tag{107.2}$$

Quick calculation confirms that these formulæ remain valid even in the cases $i = j$ and $i = k$. And as a further check on the accuracy of (106) we find (with assistance from *Mathematica*) that

$$\tfrac{1}{3!}\sum_{\text{signed permutations}} \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k = \boldsymbol{e}_{ijk} + (g\text{-terms that cancel})$$

If (as in $\mathcal{C}_2[\boldsymbol{g}]$) our indices ranged on $\{1,2\}$ then $\boldsymbol{e}_j \cdot \boldsymbol{e}_{jk}$ and $\boldsymbol{e}_{jk} \cdot \boldsymbol{e}_j$ would be essentially the only cases of interest, and we would "by descent" have

$$\begin{aligned}\boldsymbol{e}_j \cdot \boldsymbol{e}_{jk} &= -g_{jk}\boldsymbol{e}_j + g_{jj}\boldsymbol{e}_k\\ \boldsymbol{e}_{jk} \cdot \boldsymbol{e}_j &= +g_{jk}\boldsymbol{e}_j - g_{jj}\boldsymbol{e}_k\end{aligned}$$

NEXT HIGHER LEVEL To obtain the canonical development of $\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$ we use $\boldsymbol{e}_{ijkl} = \frac{1}{4}(\boldsymbol{e}_{ijk}\boldsymbol{e}_l - \boldsymbol{e}_{lij}\boldsymbol{e}_k + \boldsymbol{e}_{kli}\boldsymbol{e}_j - \boldsymbol{e}_{lij}\boldsymbol{e}_k)$ in combination with results already in hand. Looking to the details: hitting (106) with $\boldsymbol{e}_l$ on the right we get

$$\boldsymbol{e}_{ijk}\boldsymbol{e}_l = \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l - g_{ij}\boldsymbol{e}_k\boldsymbol{e}_l + g_{ki}\boldsymbol{e}_j\boldsymbol{e}_l - g_{jk}\boldsymbol{e}_i\boldsymbol{e}_l$$

whence

$$\begin{aligned}\boldsymbol{e}_{ijkl} &= \tfrac{1}{4}\sum_{\substack{\text{signed cyclic}\\ \text{permutations}}} \boldsymbol{e}_{ijk}\boldsymbol{e}_l\\ &= \tfrac{1}{4}\big\{\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l - \boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_i + \boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_i\boldsymbol{e}_j - \boldsymbol{e}_l\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\big\}\\ &\qquad + \tfrac{1}{4}\sum_{\substack{\text{signed cyclic}\\ \text{permutations}}} (-g_{ij}\boldsymbol{e}_k\boldsymbol{e}_l + g_{ki}\boldsymbol{e}_j\boldsymbol{e}_l - g_{jk}\boldsymbol{e}_i\boldsymbol{e}_l)\end{aligned}$$

But

$$\begin{aligned}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l &= \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\\ -\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_i &= \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l - 2g_{il}\boldsymbol{e}_j\boldsymbol{e}_k + 2g_{ik}\boldsymbol{e}_j\boldsymbol{e}_l - 2g_{ij}\boldsymbol{e}_k\boldsymbol{e}_l\\ \boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_i\boldsymbol{e}_j &= \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l + 2g_{il}\boldsymbol{e}_k\boldsymbol{e}_j - 2g_{ik}\boldsymbol{e}_l\boldsymbol{e}_j + 2g_{jl}\boldsymbol{e}_i\boldsymbol{e}_k - 2g_{jk}\boldsymbol{e}_i\boldsymbol{e}_l\\ -\boldsymbol{e}_l\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k &= \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l - 2g_{il}\boldsymbol{e}_j\boldsymbol{e}_k + 2g_{jl}\boldsymbol{e}_i\boldsymbol{e}_k - 2g_{kl}\boldsymbol{e}_i\boldsymbol{e}_j\end{aligned}$$

Enlisting the assistance of *Mathematica* to pull these results together, we find

$$\begin{aligned}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l = \boldsymbol{e}_{ijkl} + (g_{ij}\boldsymbol{e}_{kl} + g_{kl}\boldsymbol{e}_{ij}) - (g_{ik}\boldsymbol{e}_{jl} + g_{jl}\boldsymbol{e}_{ik}) + (g_{il}\boldsymbol{e}_{jk} + g_{jk}\boldsymbol{e}_{il})\\ + (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\,\mathsf{I}\end{aligned} \tag{108}$$

To confirm the accuracy of that statement it is sufficient to establish that the adjacent transpositional properties[22] of the expression on the right duplicate those of the expression on the left. For example, we have

$$\begin{aligned}\boldsymbol{e}_j\boldsymbol{e}_i\boldsymbol{e}_k\boldsymbol{e}_l &= -\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l + 2g_{ij}\boldsymbol{e}_k\boldsymbol{e}_l\\ &= -\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l + 2g_{ij}(\boldsymbol{e}_{kl} + g_{kl}\mathsf{I})\end{aligned}$$

which is readily seen to be mimiced by the expression on the *right* side of (107): the point to notice is that

$$\text{right side of (108)} = [ij\text{-symmetric term}] + [ij\text{-antisymmetric term}]$$

with

$$[ij\text{-symmetric term}] = g_{ij}(\boldsymbol{e}_{kl} + g_{kl}\mathsf{I})$$

As an additional check on the accuracy of (108) we have[23]

$$\sum_{\text{signed permutations}}\left\{\begin{aligned}(g_{ij}\boldsymbol{e}_{kl} + g_{kl}\boldsymbol{e}_{ij}) - (g_{ik}\boldsymbol{e}_{jl} + g_{jl}\boldsymbol{e}_{ik}) + (g_{il}\boldsymbol{e}_{jk} + g_{jk}\boldsymbol{e}_{il})\\ + (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\,\mathsf{I}\end{aligned}\right\} = 0$$

Equation (108) puts us in position canonical representations of the products $\boldsymbol{e}_i\cdot\boldsymbol{e}_{jkl}$, $\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{kl}$ and $\boldsymbol{e}_{jkl}\cdot\boldsymbol{e}_i$. We might now, with patient labor, use (108) to construct—in a moment, almost effortlessly, *will* construct—these product formulæ:

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_{jkl} = \boldsymbol{e}_{ijkl} + g_{ij}\boldsymbol{e}_{kl} + g_{ik}\boldsymbol{e}_{lj} + g_{il}\boldsymbol{e}_{jk} \tag{109.1}$$

$$\boldsymbol{e}_{jkl}\cdot\boldsymbol{e}_i = \boldsymbol{e}_{jkli} + g_{ij}\boldsymbol{e}_{kl} + g_{ik}\boldsymbol{e}_{lj} + g_{il}\boldsymbol{e}_{jk} \tag{109.2}$$

$$\begin{aligned}\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{kl} = \boldsymbol{e}_{ijkl} - (g_{ik}\boldsymbol{e}_{jl} + g_{jl}\boldsymbol{e}_{ik}) + (g_{jk}\boldsymbol{e}_{il} + g_{il}\boldsymbol{e}_{jk})\\ - (g_{ik}g_{jl} - g_{jk}g_{il})\,\mathsf{I}\end{aligned} \tag{109.3}$$

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[22] I assume my reader to be familiar with the fact that every permutation can be expressed as the product of (a characteristically even/odd number of) *transpositions of adjacent symbols*: see J. S. Lomont, *Applications of Finite Groups* (1959), page 260 or W. Burnside's classic *Theory of Groups of Finite Order* (1911), §11.

[23] Compare the "cancellations of g-terms" that were encountered on pages 32 & 33. I used resources discovered within *Mathematica*'s "Combinatorica" package to carry out the calculation, but by an improvised procedure so clumsy that the work took me nearly an hour, and that would place the case of next higher order (entails $4! \to 5!$) well beyond the limits of my patience. The time has come to acquire some computational technique!

DIGRESSION: Some *Mathematica* Technique. The properties—except for the transformation properties (weight)—that we associate with the Levi-Civita symbol $\epsilon_{i_1 i_2 \dots i_n}$ are reproduced in *Mathematica* by the command $\texttt{Signature[}\{i, j, \dots, k\}\texttt{]}$, the action of which is illustrated below:

$$\begin{aligned}\texttt{Signature[}\{1,2,3\}\texttt{]} &= +1\\ \texttt{Signature[}\{1,1,3\}\texttt{]} &= 0\\ \texttt{Signature[}\{2,1,3\}\texttt{]} &= -1\end{aligned}$$

The command is powerful enough to read subscripts, thus

$$\begin{aligned}\texttt{Signature[}\{\alpha_1,\alpha_2,\alpha_3\}\texttt{]} &= +1\\ \texttt{Signature[}\{\alpha_1,\alpha_1,\alpha_3\}\texttt{]} &= 0\\ \texttt{Signature[}\{\alpha_2,\alpha_1,\alpha_3\}\texttt{]} &= -1\end{aligned}$$

This permits us to use subscripts to orchestrate sums of the sort in which the Levi-Civita symbol is a frequent participant:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]F[}\alpha_i,\alpha_j\texttt{]} = \texttt{F[}\alpha_1,\alpha_2\texttt{]}-\texttt{F[}\alpha_2,\alpha_1\texttt{]}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]}\texttt{e}_{i,j} = \texttt{e}_{1,2}-\texttt{e}_{2,1}$$

We must, however, be prepared to work around the fact that *Mathematica*'s natural instinct is to assume commutivity:

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]}\texttt{e}_i\texttt{e}_j &= 0\\ &\neq \texttt{e}_1\texttt{e}_2 - \texttt{e}_2\texttt{e}_1\end{aligned}$$

Here I use the technique described above to reconstruct the definition of determinant:

$$\det\begin{pmatrix} g_{1,1} & g_{1,2}\\ g_{2,1} & g_{2,2}\end{pmatrix} - \sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]}g_{1,i}g_{2,j} = 0$$

The commas are, by the way, critical, for in their absence *Mathematica* would *multiply* the subscripts, as demonstrated below:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]}g_{i,j} = g_{1,2} - g_{2,1}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]}\ g_{ij} = g_{1\cdot 2} - g_{2\cdot 1} = 0$$

Finally—in anticipation of things to come—I use the technique to reproduce the derivation of (107.1) from (106). Let the latter identity

$$\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k = \boldsymbol{e}_{ijk} + g_{ij}\boldsymbol{e}_k - g_{ki}\boldsymbol{e}_j + g_{jk}\boldsymbol{e}_i$$

be notated

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_{j_1}\boldsymbol{e}_{j_2} = F(i, j_1, j_2)$$

where

$$\begin{aligned} F(i,j,k) \equiv \boldsymbol{e}_{i,j,k} &+ \tfrac{1}{2}(g_{i,j} + g_{j,i})\boldsymbol{e}_k \\ &- \tfrac{1}{2}(g_{k,i} + g_{i,k})\boldsymbol{e}_j + \tfrac{1}{2}(g_{j,k} + g_{k,j})\boldsymbol{e}_i \end{aligned}$$

has been spelled out in such a way as (in effect) to inform *Mathematica* that $g_{ij} = g_{ji}$. Thus prepared, we write

$$\boldsymbol{e}_m \cdot \boldsymbol{e}_{n_1 n_2} = \tfrac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\texttt{Signature[}\{i,j\}\texttt{]}\,F(m, n_i, n_j)$$

and instantly recover(107). Note also that we are in position now to reestablish—this time without labor—the g-independence of

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\texttt{Signature[}\{i,j,k\}\texttt{]}\,F(n_i, n_j, n_k)$$

To prepare for application of those techniques to the derivation of the product formulæ (109) we define $F(i,j,k,l)$ by making substitutions

$$g_{mn} \longmapsto \frac{g_{m,n} + g_{n,m}}{2}\,, \quad \boldsymbol{e}_{ij} \longmapsto e_{i,j}\,, \quad \boldsymbol{e}_{ijkl} \longmapsto e_{i,j,k,l}$$

into the expression that appears on the right side of (108). Then, to obtain (109.1), we evaluate

$$\tfrac{1}{3!}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\texttt{Signature[}\{j,k,l\}\texttt{]}\,F(i,j,k,l)$$

and make notational adjustments $1 \mapsto j$, $2 \mapsto k$, $3 \mapsto l$. To obtain (109.2) we evaluate

$$\tfrac{1}{3!}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\texttt{Signature[}\{j,k,l\}\texttt{]}\,F(j,k,l,i)$$

and proceed similarly. To obtain (109.3) we evaluate

$$\tfrac{1}{2!2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=3}^{4}\sum_{l=3}^{4}\texttt{Signature[}\{i,j\}\texttt{]}\;\texttt{Signature[}\{k,l\}\texttt{]}\,F(i,j,k,l)$$

and make notational adjustments $1 \mapsto 1$, $2 \mapsto j$, $3 \mapsto k$, $4 \mapsto l$.

But if general multiplication (inversion, similarity transformation, *etc.*) within $\mathcal{C}_3[\boldsymbol{g}]$ is our objective then our work is not yet done: we must

1) develop the canonical representation of $\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m$ and use that information (and the preceding techniques) to construct descriptions of
   - $\boldsymbol{e}_i \cdot \boldsymbol{e}_{jklm}$ (not actually needed until we come to $\mathcal{C}_4[\boldsymbol{g}]$)
   - $\boldsymbol{e}_{ij} \cdot \boldsymbol{e}_{klm}$
   - $\boldsymbol{e}_{klm} \cdot \boldsymbol{e}_{ij}$
   - $\boldsymbol{e}_{jklm} \cdot \boldsymbol{e}_i$ (not actually needed until we come to $\mathcal{C}_4[\boldsymbol{g}]$)
2) develop the canonical representation of $\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n$ and use that information…to construct descriptions of
   - $\boldsymbol{e}_i \cdot \boldsymbol{e}_{jklmn}$ (not actually needed until we come to $\mathcal{C}_5[\boldsymbol{g}]$)
   - $\boldsymbol{e}_{ij} \cdot \boldsymbol{e}_{klmn}$ (not actually needed until we come to $\mathcal{C}_4[\boldsymbol{g}]$)
   - $\boldsymbol{e}_{ijk} \cdot \boldsymbol{e}_{lmn}$
   - $\boldsymbol{e}_{klmn} \cdot \boldsymbol{e}_{ij}$ (not actually needed until we come to $\mathcal{C}_4[\boldsymbol{g}]$)
   - $\boldsymbol{e}_{jklmn} \cdot \boldsymbol{e}_i$ (not actually needed until we come to $\mathcal{C}_5[\boldsymbol{g}]$)

Note that, because of the recursive design of the theory (calculations in any specified order make essential use of lower order results), those problems must be approached in the order stated.

NEXT HIGHER LEVEL We proceed in imitation of the pattern established on page 33. Noting that the cyclic permutations of $\{i, j, k, l, m\}$ are all even, we have

$$\begin{aligned}\boldsymbol{e}_{ijklm} = \tfrac{1}{5}(\boldsymbol{e}_{ijkl}\boldsymbol{e}_m + \boldsymbol{e}_{mijk}\boldsymbol{e}_l + \boldsymbol{e}_{lmij}\boldsymbol{e}_k + \boldsymbol{e}_{klmi}\boldsymbol{e}_j + \boldsymbol{e}_{jklm}\boldsymbol{e}_i)&\\ \boldsymbol{e}_{ijkl}\boldsymbol{e}_m = \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m &- (g_{ij}\boldsymbol{e}_{kl} + g_{kl}\boldsymbol{e}_{ij})\boldsymbol{e}_m\\ &+ (g_{ik}\boldsymbol{e}_{jl} + g_{jl}\boldsymbol{e}_{ik})\boldsymbol{e}_m\\ &- (g_{il}\boldsymbol{e}_{jk} + g_{jk}\boldsymbol{e}_{il})\boldsymbol{e}_m\\ &- (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\boldsymbol{e}_m \text{ by (108)}\end{aligned}$$

giving

$$\begin{aligned}\boldsymbol{e}_{ijklm} = \tfrac{1}{5}\big\{\!\big\{\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m + \boldsymbol{e}_m\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l + \boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k + \boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_i\big\}\!\big\}&\\ + \tfrac{1}{5}\sum_{\substack{\text{cyclic}\\ \text{permutations}}} \Big\{ &- (g_{ij}\boldsymbol{e}_{kl} + g_{kl}\boldsymbol{e}_{ij})\boldsymbol{e}_m\\ &+ (g_{ik}\boldsymbol{e}_{jl} + g_{jl}\boldsymbol{e}_{ik})\boldsymbol{e}_m\\ &- (g_{il}\boldsymbol{e}_{jk} + g_{jk}\boldsymbol{e}_{il})\boldsymbol{e}_m\\ &- (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\boldsymbol{e}_m\Big\}\end{aligned}$$

A computation as tedious as it is elementary (it makes use only of the basic identity $\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i + 2g_{ij}\,\mathbf{I}$) supplies

$$\{\!\{ etc. \}\!\} = 5\,\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m + 2(g_{im}\boldsymbol{p}_{jkl} - g_{jm}\boldsymbol{p}_{ikl} + g_{km}\boldsymbol{p}_{ijl} - g_{lm}\boldsymbol{p}_{ijk})$$
$$+\,2(g_{im}\boldsymbol{p}_{ljk} - g_{jm}\boldsymbol{p}_{lik} + g_{km}\boldsymbol{p}_{lij} - g_{il}\boldsymbol{p}_{jkm} + g_{jl}\boldsymbol{p}_{ikm} - g_{kl}\boldsymbol{p}_{ijm})$$
$$+\,2(g_{im}\boldsymbol{p}_{klj} - g_{il}\boldsymbol{p}_{kmj} + g_{ik}\boldsymbol{p}_{lmj} - g_{jm}\boldsymbol{p}_{ikl} + g_{jl}\boldsymbol{p}_{ikm} - g_{jk}\boldsymbol{p}_{ilm})$$
$$+\,2(g_{im}\boldsymbol{p}_{jkl} - g_{il}\boldsymbol{p}_{jkm} + g_{ik}\boldsymbol{p}_{jlm} - g_{ij}\boldsymbol{p}_{klm})$$

with

$$\begin{aligned}\boldsymbol{p}_{ijk} &\equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\\ &= \boldsymbol{e}_{ijk} + g_{ij}\boldsymbol{e}_k - g_{ki}\boldsymbol{e}_j + g_{jk}\boldsymbol{e}_i\end{aligned}$$

We are in position now to consign all remaining computational tedium to *Mathematica*. To that end, we enter these definitions[24]

$$\texttt{G[i\_,j\_]:} = \frac{g_{i,j} + g_{j,i}}{2}$$

$$\texttt{p[i\_,j\_,k\_]:} = e_{i,j,k} + G[i,j]e_k - G[k,i]e_j + G[j,k]e_i$$

$$\texttt{q[i\_,j\_,k\_]:} = e_{i,j,k} + G[k,j]e_i - G[k,i]e_j$$

$$\begin{aligned}\texttt{A[i\_,j\_,k\_,l\_,m\_]:} = {}&5e_{i,j,k,l,m} + 2(G[i,m,]p[j,k,l] - \cdots)\\ &\vdots\\ &+2(\cdots - G[i,j]p[k,l,m])\end{aligned}$$

$$\begin{aligned}\texttt{B[i\_,j\_,k\_,l\_,m\_]:} = {}&-(G[i,j]\,q[k,l,m] + G[k,l]\,q[i,j,m])\\ &+(G[i,k]\,q[j,l,m] + G[j,l]\,q[i,k,m])\\ &-(G[i,l]\,q[j,k,m] + G[j,k]\,q[i,l,m])\\ &+(G[i,j]G[k,l] - G[i,k]G[j,l] + G[i,l]G[i,k])e_m\end{aligned}$$

and ask for the evaluation of

$$\tfrac{1}{5}A[i,j,k,l,m] + \tfrac{1}{5}\big\{\quad B[i,j,k,l,m] + B[m,i,j,k,l]\big\}$$
$$+\,B[l,m,i,j,k] + B[k,l,m,i,j] + B[j,k,l,m,i]$$

Mathematica promptly disgorges a flood of output: our non-trivial assignment is to sort though it, make patterned sense of it. Thus am I brought at length to the canonical decomposition of $\boldsymbol{p}_{ijklm} \equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m$ that is presented as equation (110) on the next page. I will not comment explicitly on the sign distribution, except to remark that it appears on its face to be semi-intelligible.[25]

[24] The construction of `p[i,j,k]` reflects the description (106) of $\boldsymbol{p}_{ijk} \equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k$, `q[i,j,k]` reflects the description (17.2) of $\boldsymbol{e}_{ij}\boldsymbol{e}_k$. The definitions `A[i,j,k,l,m]` and `B[i,j,k,l,m]` are motivated by the design of the final equation on the preceding page.

[25] We have reached a point at which typographic accuracy has become a major consideration, and where by-hand simplification—even though actually done on-screen—has become hazardous.

$$
\begin{aligned}
\boldsymbol{p}_{ijklm} \equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m = \boldsymbol{e}_{ijklm} &+ g_{ij}\boldsymbol{e}_{klm} \\
&- g_{ik}\boldsymbol{e}_{jlm} \\
&+ g_{il}\boldsymbol{e}_{jkm} \\
&- g_{im}\boldsymbol{e}_{jkl} \\
&+ g_{jk}\boldsymbol{e}_{ilm} \\
&- g_{jl}\boldsymbol{e}_{ikm} \\
&+ g_{jm}\boldsymbol{e}_{ikl} \\
&+ g_{kl}\boldsymbol{e}_{ijm} \\
&- g_{km}\boldsymbol{e}_{ijl} \\
&+ g_{lm}\boldsymbol{e}_{ijk} \\
&+ (g_{jk}g_{lm} - g_{jl}g_{km} + g_{jm}g_{kl})\,\boldsymbol{e}_i \\
&- (g_{kl}g_{mi} - g_{km}g_{li} + g_{ki}g_{lm})\,\boldsymbol{e}_j \\
&+ (g_{lm}g_{ij} - g_{li}g_{mj} + g_{lj}g_{mi})\,\boldsymbol{e}_k \\
&- (g_{mi}g_{jk} - g_{mj}g_{ik} + g_{mk}g_{ij})\,\boldsymbol{e}_l \\
&+ (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\,\boldsymbol{e}_m
\end{aligned}
\tag{110}
$$

On this basis we carefully enter into our *Mathematica* notebook the definition

$$
\begin{aligned}
\texttt{F[i\_,j\_,k\_,l\_,m\_]:=}\; e_{i,j,k,l,m} &+ G[i,j]e_{k,l,m} \\
&- G[i,k]e_{j,l,m} \\
&\;\vdots \\
&+ (G[i,j]G[k,l] - \cdots + G[i,l]G[j,k])e_m
\end{aligned}
$$

As checks on the accuracy of (110) we observe, for example, that

$$
\boldsymbol{p}_{ijkmm} = \boldsymbol{p}_{ijk}\cdot g_{mm}
$$

while

$$
\begin{aligned}
F(i,j,k,m,m) &= \big\{\boldsymbol{e}_{ijk} + g_{ij}\boldsymbol{e}_k - g_{ki}\boldsymbol{e}_j + g_{jk}\boldsymbol{e}_i\big\}\cdot g_{mm} \\
&= \boldsymbol{p}_{ijk}\cdot g_{mm} \quad \text{by (106)}
\end{aligned}
$$

—the interesting point here being that high-order formulæ can be used to generate/reproduce lower-order formulæ (the catch being that the latter are needed to *derive* the former!). We also find that all the g-terms disappear from

$$
\tfrac{1}{5!}\sum_{i,j,k,l,m=1}^{5}\texttt{Signature[}i,j,k,l,m\texttt{]}F(i,j,k,l,m)
$$

—leaving us with what is, in fact, precisely the *definition* of $\boldsymbol{e}_{ijklm}$. Further checks on the accuracy of (110) and of the transcription of $F(i,j,k,l,m)$ into our notebook are provided by

$$F(i,j,k,m,m) = F(i,j,m,m,k) = F(i,m,m,j,k) = F(m,m,i,j,k)$$

and

$$F(i,j,j,k,k) = F(j,j,i,k,k) = F(j,j,k,k,i)$$

Satisfied that all is correct,[26] we ask *Mathematica* to construct

$$\tfrac{1}{1!4!}\sum_{j,k,l,m=2}^{5} \texttt{Signature[}j,k,l,m\texttt{]}F(1,j,k,l,m)$$

$$\tfrac{1}{2!3!}\sum_{i,j=1}^{2}\sum_{k,l,m=3}^{5} \texttt{Signature[}i,j\texttt{]Signature[}k,l,m\texttt{]}F(i,j,k,l,m)$$

$$\tfrac{1}{3!2!}\sum_{k,l,m=3}^{5}\sum_{i,j=1}^{2} \texttt{Signature[}k,l,m\texttt{]Signature[}i,j\texttt{]}F(k,l,m,i,j)$$

$$\tfrac{1}{4!1!}\sum_{j,k,l,m=2}^{5} \texttt{Signature[}j,k,l,m\texttt{]}F(j,k,l,m,1)$$

and, by notational adjustment of its output, obtain

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_{jklm} = \boldsymbol{e}_{ijklm} + (g_{ij}\boldsymbol{e}_{klm} - g_{ik}\boldsymbol{e}_{jlm} + g_{il}\boldsymbol{e}_{jkm} - g_{im}\boldsymbol{e}_{jkl}) \tag{111.1}$$

$$\begin{aligned}\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{klm} = \boldsymbol{e}_{ijklm} &- (g_{ik}\boldsymbol{e}_{jlm} - g_{il}\boldsymbol{e}_{jkm} + g_{im}\boldsymbol{e}_{jkl})\\ &+ (g_{jk}\boldsymbol{e}_{ilm} - g_{jl}\boldsymbol{e}_{ikm} + g_{jm}\boldsymbol{e}_{ikl})\\ &- (g_{il}g_{jm} - g_{im}g_{jl})\boldsymbol{e}_k\\ &- (g_{im}g_{jk} - g_{ik}g_{jm})\boldsymbol{e}_l\\ &- (g_{ik}g_{jl} - g_{il}g_{jk})\boldsymbol{e}_m\end{aligned} \tag{111.2}$$

$$\begin{aligned}\boldsymbol{e}_{klm}\cdot\boldsymbol{e}_{ij} = \boldsymbol{e}_{ijklm} &+ (g_{ik}\boldsymbol{e}_{jlm} - g_{il}\boldsymbol{e}_{jkm} + g_{im}\boldsymbol{e}_{jkl})\\ &- (g_{jk}\boldsymbol{e}_{ilm} - g_{jl}\boldsymbol{e}_{ikm} + g_{jm}\boldsymbol{e}_{ikl})\\ &- (g_{il}g_{jm} - g_{im}g_{jl})\boldsymbol{e}_k\\ &- (g_{im}g_{jk} - g_{ik}g_{jm})\boldsymbol{e}_l\\ &- (g_{ik}g_{jl} - g_{il}g_{jk})\boldsymbol{e}_m\end{aligned} \tag{111.3}$$

$$\boldsymbol{e}_{jklm}\cdot\boldsymbol{e}_i = \boldsymbol{e}_{ijklm} - (g_{ij}\boldsymbol{e}_{klm} - g_{ik}\boldsymbol{e}_{jlm} + g_{il}\boldsymbol{e}_{jkm} - g_{im}\boldsymbol{e}_{jkl}) \tag{111.4}$$

To test—if only weakly—the accuracy of the preceding formulæ we might look in the Euclidean case to such products as $\boldsymbol{e}_1\cdot\boldsymbol{e}_{2345}$ and $\boldsymbol{e}_2\cdot\boldsymbol{e}_{2345}$.

[26] It is of critical importance that everything be *precisely* correct, for errors at any given order propagate to all higher orders.

NEXT HIGHER LEVEL As was remarked already on page 37, we must develop the canonical representation of $\boldsymbol{p}_{ijklmn} \equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n$ (whence, in particular, of $\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmn}$) before we will be in position to work out the theory of $\mathcal{C}_3[\boldsymbol{g}]$. We proceed from

$$\boldsymbol{e}_{ijklmn} = \tfrac{1}{6}(\boldsymbol{e}_{ijklm}\boldsymbol{e}_n - \boldsymbol{e}_{nijkl}\boldsymbol{e}_m + \boldsymbol{e}_{mnijk}\boldsymbol{e}_l - \boldsymbol{e}_{lmnij}\boldsymbol{e}_k + \boldsymbol{e}_{klmni}\boldsymbol{e}_j - \boldsymbol{e}_{jklmn}\boldsymbol{e}_i)$$

$$\begin{aligned}
\boldsymbol{e}_{ijklm}\boldsymbol{e}_n = \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n &- g_{ij}\boldsymbol{e}_{klm}\boldsymbol{e}_n\\
&+ g_{ik}\boldsymbol{e}_{jlm}\boldsymbol{e}_n\\
&- g_{il}\boldsymbol{e}_{jkm}\boldsymbol{e}_n\\
&+ g_{im}\boldsymbol{e}_{jkl}\boldsymbol{e}_n\\
&- g_{jk}\boldsymbol{e}_{ilm}\boldsymbol{e}_n\\
&+ g_{jl}\boldsymbol{e}_{ikm}\boldsymbol{e}_n\\
&- g_{jm}\boldsymbol{e}_{ikl}\boldsymbol{e}_n\\
&- g_{kl}\boldsymbol{e}_{ijm}\boldsymbol{e}_n\\
&+ g_{km}\boldsymbol{e}_{ijl}\boldsymbol{e}_n\\
&- g_{lm}\boldsymbol{e}_{ijk}\boldsymbol{e}_n\\
&- (g_{jk}g_{lm} - g_{jl}g_{km} + g_{jm}g_{kl})\,\boldsymbol{e}_i\boldsymbol{e}_n\\
&+ (g_{kl}g_{mi} - g_{km}g_{li} + g_{ki}g_{lm})\,\boldsymbol{e}_j\boldsymbol{e}_n\\
&- (g_{lm}g_{ij} - g_{li}g_{mj} + g_{lj}g_{mi})\,\boldsymbol{e}_k\boldsymbol{e}_n\\
&+ (g_{mi}g_{jk} - g_{mj}g_{ik} + g_{mk}g_{ij})\,\boldsymbol{e}_l\boldsymbol{e}_n\\
&- (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\,\boldsymbol{e}_m\boldsymbol{e}_n\\
\equiv \boldsymbol{p}_{ijklmn} - B_{ijklmn}&
\end{aligned}$$

Introducing the abbreviation

$$\begin{aligned}
a_{ijklmn} &\equiv g_{ij}\,\boldsymbol{p}_{klmn}\\
\boldsymbol{p}_{klmn} &\equiv \boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n \text{ developed at (108)}
\end{aligned}$$

we find by careful pencil-&-paper work that

$$\begin{aligned}
&\boldsymbol{p}_{ijklmn} - \boldsymbol{p}_{nijklm} + \boldsymbol{p}_{mnijkl} - \boldsymbol{p}_{lmnijk} + \boldsymbol{p}_{klmnij} - \boldsymbol{p}_{jklmni}\\
&\quad = 6\,\boldsymbol{p}_{ijklmn} + 2(-a_{injklm} + a_{jniklm} - a_{knijlm} + a_{lnijkm} - a_{mnijkl})\\
&\qquad + 2(\ \ a_{inmjkl} - a_{jnmikl} + a_{knmijl} - a_{lnmijk}\\
&\qquad\qquad + a_{imjkln} - a_{jmikln} + a_{kmijln} - a_{lmijkn})\\
&\qquad + 2(-a_{inlmjk} + a_{jnlmik} - a_{knlmij} + a_{imljkn}\\
&\qquad\qquad - a_{jmlikn} + a_{kmlijn} - a_{iljkmn} + a_{jlikmn} - a_{klijmn})\\
&\qquad + 2(\ \ a_{inklmj} - a_{jnklmi} + a_{imkljn} - a_{jmklin}\\
&\qquad\qquad + a_{ilkjmn} - a_{jlkimn} + a_{ikjlmn} - a_{jkilmn})\\
&\qquad + 2(-a_{injklm} + a_{imjkln} - a_{iljkmn} + a_{ikjlmn} - a_{ijklmn})\\
&\quad \equiv 6\,\boldsymbol{p}_{ijklmn} - A_{ijklmn}
\end{aligned}$$

Assembly of those results gives

$$
\begin{aligned}
\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n = \boldsymbol{e}_{ijklmn} &+ \tfrac{1}{6}A_{ijklmn} \\
&+ \tfrac{1}{6}\big\{B_{ijklmn} - B_{nijklm} + B_{mnijkl} \\
&\qquad - B_{lmnijk} + B_{klmnij} - B_{jklmni}\big\}
\end{aligned}
\tag{112}
$$

This equation describes $\boldsymbol{p}_{ijklmn} \equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n$ as a linear combination of $\boldsymbol{e}_{ijklmn}$, $g_{ij}\,\boldsymbol{e}_{klm}\cdot\boldsymbol{e}_n$, $g_{ij}g_{kl}\,\boldsymbol{e}_m\cdot\boldsymbol{e}_n$ and $g_{ij}\,\boldsymbol{p}_{klmn}$-type terms. Using (109.2), (105) and (108) to describe the canonical representations of $\boldsymbol{e}_m\cdot\boldsymbol{e}_n$, $\boldsymbol{e}_{klm}\cdot\boldsymbol{e}_n$ and $\boldsymbol{p}_{klmn}$, we carefully feed the right side of (112) into *Mathematica* (took me the better part of an hour) and in a few seconds obtain an enormously (!) long string of $g_{ij}\,\boldsymbol{e}_{klmn}$, $g_{ij}g_{kl}\,\boldsymbol{e}_{mn}$ and $g_{ij}g_{kl}g_{mn}\mathbf{I}$ terms (plus a solitary $\boldsymbol{e}_{ijklmn}$). Carefully exploiting the symmetry of g_{ij} and the total antisymmetry of $\boldsymbol{e}_{ij}$, $\boldsymbol{e}_{ijkl}$ to consolidate those terms, I at length (which is to say: after a long afternoon's work) obtained

$$
\begin{aligned}
\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m\boldsymbol{e}_n = \boldsymbol{e}_{ijklmn} &+ g_{ij}\,\boldsymbol{e}_{klmn} + \boldsymbol{e}_{ij}\,g_{klmn} + g_{ij}\,g_{klmn}\mathbf{I} \\
&- g_{ik}\,\boldsymbol{e}_{jlmn} - \boldsymbol{e}_{ik}\,g_{jlmn} - g_{ik}\,g_{jlmn}\mathbf{I} \\
&+ g_{il}\,\boldsymbol{e}_{jkmn} + \boldsymbol{e}_{il}\,g_{jkmn} + g_{il}\,g_{jkmn}\mathbf{I} \\
&- g_{im}\,\boldsymbol{e}_{jkln} - \boldsymbol{e}_{im}\,g_{jkln} - g_{im}\,g_{jkln}\mathbf{I} \\
&+ g_{in}\,\boldsymbol{e}_{jklm} + \boldsymbol{e}_{in}\,g_{jklm} + g_{in}\,g_{jklm}\mathbf{I} \\
&+ g_{jk}\,\boldsymbol{e}_{ilmn} + \boldsymbol{e}_{jk}\,g_{ilmn} \\
&- g_{jl}\,\boldsymbol{e}_{ikmn} - \boldsymbol{e}_{jl}\,g_{ikmn} \\
&+ g_{jm}\,\boldsymbol{e}_{ikln} + \boldsymbol{e}_{jm}\,g_{ikln} \\
&- g_{jn}\,\boldsymbol{e}_{iklm} - \boldsymbol{e}_{jn}\,g_{iklm} \\
&+ g_{kl}\,\boldsymbol{e}_{ijmn} + \boldsymbol{e}_{kl}\,g_{ijmn} \\
&- g_{km}\,\boldsymbol{e}_{ijln} - \boldsymbol{e}_{km}\,g_{ijln} \\
&+ g_{kn}\,\boldsymbol{e}_{ijlm} + \boldsymbol{e}_{kn}\,g_{ijlm} \\
&+ g_{lm}\,\boldsymbol{e}_{ijkn} + \boldsymbol{e}_{lm}\,g_{ijkn} \\
&- g_{ln}\,\boldsymbol{e}_{ijkm} - \boldsymbol{e}_{ln}\,g_{ijkm} \\
&+ g_{mn}\,\boldsymbol{e}_{ijkl} + \boldsymbol{e}_{mn}\,g_{ijkl} \\
\equiv F(i,j,k,l,m,n)&
\end{aligned}
\tag{113}
$$

with

$$
g_{ijmn} \equiv g_{ij}g_{mn} - g_{im}g_{jn} + g_{in}g_{jm} \tag{114}
$$

We note that the signs are precisely those that would result if subscripts were introduced into $\epsilon_{\cdots\cdots}$ and then brought to standard $ijklmn$ order. Informing

Mathematica of the definition of $F(i,j,k,l,m,n)$, we first construct[27]

$$\tfrac{1}{6!}\sum_{i,j,k,l,m,n=1}^{6}\texttt{Signature[}\{i,j,k,l,m,n\}\texttt{]}\,F(i,j,k,l,m,n)$$

and obtain a string of the $6! = 720$ signed permutations of $e_{1,2,3,4,5,6}$ from which all reference to g_{ij} has vanished—leaving us (compare page 39) with what is, in fact, precisely the *definition* of $\boldsymbol{e}_{ijklmn}$. This I take to be strong evidence that (113) is correct, and has been accurately transcribed into our *Mathematica* notebook. Further evidence is provided—here as on pages 39 & 40—by verified statements of the types

$$F(i,j,k,l,m,m) = F(m,m,i,j,k,l), \;\; \textit{etc.}$$

$$F(i,j,m,m,n,n) = (\boldsymbol{e}_{ij} + g_{ij}\mathbf{I})g_{mm}g_{nn}$$

We are in position now to construct canonical developments of all five of the sixth-order products listed on page 37. I will concern myself, however, only with the product $\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmn}$ that is directly relevant to the theory of $\mathcal{C}_3[\boldsymbol{g}]$. From the reported value of

$$\tfrac{1}{3!3!}\sum_{i,j,k=1}^{3}\sum_{l,m,n=4}^{6}\texttt{Signature[}\{i,j,k\}\texttt{]}\texttt{Signature[}\{i,j,k\}\texttt{]}\,F(i,j,k,l,m,n)$$

we extract

$$\begin{aligned}
\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmn} = \boldsymbol{e}_{ijklmn} &+ g_{il}\,\boldsymbol{e}_{jkmn} - \boldsymbol{e}_{il}\,(g_{jm}g_{kn} - g_{jn}g_{km})\\
&- g_{im}\,\boldsymbol{e}_{jkln} + \boldsymbol{e}_{im}\,(g_{jl}g_{kn} - g_{jn}g_{kl})\\
&+ g_{in}\,\boldsymbol{e}_{jklm} - \boldsymbol{e}_{in}\,(g_{jl}g_{km} - g_{jm}g_{kl})\\
&- g_{jl}\,\boldsymbol{e}_{ikmn} + \boldsymbol{e}_{jl}\,(g_{im}g_{kn} - g_{in}g_{km})\\
&+ g_{jm}\,\boldsymbol{e}_{ikln} - \boldsymbol{e}_{jm}\,(g_{il}g_{kn} - g_{in}g_{kl})\\
&- g_{jn}\,\boldsymbol{e}_{iklm} + \boldsymbol{e}_{jn}\,(g_{il}g_{km} - g_{im}g_{kl})\\
&+ g_{kl}\,\boldsymbol{e}_{ijmn} - \boldsymbol{e}_{kl}\,(g_{im}g_{jn} - g_{in}g_{jm})\\
&- g_{km}\,\boldsymbol{e}_{ijln} + \boldsymbol{e}_{km}\,(g_{il}g_{jn} - g_{in}g_{jl})\\
&+ g_{kn}\,\boldsymbol{e}_{ijlm} - \boldsymbol{e}_{kn}\,(g_{il}g_{jm} - g_{im}g_{jl}) + g_{ikjlmn}\,\mathbf{I}
\end{aligned}\tag{115.1}$$

where $g_{ijklmn} = \frac{1}{8}$(eight permuted copies of six terms) and can, we notice, be described

$$g_{ijklmn} = -\det\begin{pmatrix} g_{il} & g_{im} & g_{in}\\ g_{jl} & g_{jm} & g_{jn}\\ g_{kl} & g_{km} & g_{kn}\end{pmatrix}\tag{116}$$

[27] No small assignnment, this: it took *Mathematica 5*, running at 1.6 GHz on my PowerMac G5, 145.34 seconds to accomplish the feat, and required 18.3 MB of memory.

Similar calculations supply

$$\left.\begin{matrix}\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{klmn}\\ \boldsymbol{e}_{klmn}\cdot\boldsymbol{e}_{ij}\end{matrix}\right\} = \boldsymbol{e}_{ijklmn} \mp g_{ik}\boldsymbol{e}_{jlmn} \pm g_{il}\boldsymbol{e}_{jkmn} \mp g_{im}\boldsymbol{e}_{jkln} \pm g_{in}\boldsymbol{e}_{jklm}$$
$$\begin{aligned}&\pm g_{jk}\boldsymbol{e}_{ilmn} \mp g_{jl}\boldsymbol{e}_{ikmn} \pm g_{jm}\boldsymbol{e}_{ikln} \mp g_{jn}\boldsymbol{e}_{iklm}\\ &-(g_{im}g_{jn}-g_{in}g_{jm})\boldsymbol{e}_{kl}\\ &+(g_{il}\,g_{jn}-g_{in}\,g_{jl}\,)\boldsymbol{e}_{km}\\ &-(g_{il}\,g_{jm}-g_{im}g_{jl}\,)\boldsymbol{e}_{kn}\\ &-(g_{ik}\,g_{jn}-g_{in}\,g_{jk}\,)\boldsymbol{e}_{lm}\\ &+(g_{ik}g_{jm}-g_{im}g_{jk}\,)\boldsymbol{e}_{ln}\\ &-(g_{ik}\,g_{jl}\,-g_{il}\,g_{jk}\,)\boldsymbol{e}_{mn}\end{aligned} \tag{115.2}$$

We are inspired by the point remarked at (116) to observe that if we proceed from

$$\begin{pmatrix} g_{il} & g_{im} & g_{in}\\ g_{jl} & g_{jm} & g_{jn}\\ g_{kl} & g_{km} & g_{kn}\end{pmatrix}$$

to the associated matrix of cofactors (or "signed minors")

$$\begin{pmatrix} G^{il} & G^{im} & G^{in}\\ G^{jl} & G^{jm} & G^{jn}\\ G^{kl} & G^{km} & G^{kn}\end{pmatrix} \equiv \begin{pmatrix} +\begin{vmatrix} g_{jm} & g_{jn}\\ g_{km} & g_{kn}\end{vmatrix} & -\begin{vmatrix} g_{jl} & g_{jn}\\ g_{kl} & g_{kn}\end{vmatrix} & +\begin{vmatrix} g_{jl} & g_{jm}\\ g_{kl} & g_{km}\end{vmatrix}\\[2ex] -\begin{vmatrix} g_{im} & g_{in}\\ g_{km} & g_{kn}\end{vmatrix} & +\begin{vmatrix} g_{il} & g_{in}\\ g_{kl} & g_{kn}\end{vmatrix} & -\begin{vmatrix} g_{il} & g_{im}\\ g_{kl} & g_{km}\end{vmatrix}\\[2ex] +\begin{vmatrix} g_{im} & g_{in}\\ g_{jm} & g_{jn}\end{vmatrix} & -\begin{vmatrix} g_{il} & g_{in}\\ g_{jl} & g_{jn}\end{vmatrix} & +\begin{vmatrix} g_{il} & g_{im}\\ g_{jl} & g_{jm}\end{vmatrix}\end{pmatrix}$$

then (115.1) assumes the more orderly form

$$\begin{aligned}\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmn} = \boldsymbol{e}_{ijklmn} &+ g_{il}\,\boldsymbol{e}_{jkmn} - \boldsymbol{e}_{il}\,G^{il}\\ &- g_{im}\,\boldsymbol{e}_{jkln} - \boldsymbol{e}_{im}\,G^{im}\\ &+ g_{in}\,\boldsymbol{e}_{jklm} - \boldsymbol{e}_{in}\,G^{in}\\ &- g_{jl}\,\boldsymbol{e}_{ikmn} - \boldsymbol{e}_{jl}\,G^{jl}\\ &+ g_{jm}\,\boldsymbol{e}_{ikln} - \boldsymbol{e}_{jm}\,G^{jm}\\ &- g_{jn}\,\boldsymbol{e}_{iklm} - \boldsymbol{e}_{jn}\,G^{jn}\\ &+ g_{kl}\,\boldsymbol{e}_{ijmn} - \boldsymbol{e}_{kl}\,G^{kl}\\ &- g_{km}\,\boldsymbol{e}_{ijln} - \boldsymbol{e}_{km}\,G^{km}\\ &+ g_{kn}\,\boldsymbol{e}_{ijlm} - \boldsymbol{e}_{kn}\,G^{kn} - \det\begin{pmatrix} g_{il} & g_{im} & g_{in}\\ g_{jl} & g_{jm} & g_{jn}\\ g_{kl} & g_{km} & g_{kn}\end{pmatrix}\mathbf{I}\end{aligned} \tag{117.1}$$

and that similar notational simplifications can be brought to (115.2). Look in this light back to the description (109.3) of $\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{kl}$, which if we proceed

$$\begin{pmatrix} g_{ik} & g_{il} \\ g_{jk} & g_{jl} \end{pmatrix} \longmapsto \text{matrix of cofactors } \begin{pmatrix} G^{ik} & G^{il} \\ G^{jk} & G^{jl} \end{pmatrix} = \begin{pmatrix} +g_{jl} & -g_{jk} \\ -g_{il} & +g_{ik} \end{pmatrix}$$

can be cast into the form

$$\begin{aligned} \boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{kl} = \boldsymbol{e}_{ijkl} &- \boldsymbol{e}_{ik}\,G^{ik} \\ &- \boldsymbol{e}_{il}\,G^{il} \\ &- \boldsymbol{e}_{jk}\,G^{jk} \\ &- \boldsymbol{e}_{jl}\,G^{jl} - \det\begin{pmatrix} g_{ik} & g_{il} \\ g_{jk} & g_{jl} \end{pmatrix}\mathbf{I} \end{aligned} \tag{117.2}$$

As a check on the accuracy of (115) we verify that (non-obviously) the expression on the right possesses both the $\{i,j,k\}$-antisymmetry and the $\{l,m,n\}$-antisymmetry that are manifest on the left.

In $\mathcal{C}_3[\mathbb{g}]$ all indices range on $\{1,2,3\}$. It follows in that instance that

$$\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmn} = \pm\boldsymbol{e}_{123}\cdot\boldsymbol{e}_{123} \quad \text{if it does not vanish}$$

And from (115) it follows by quick calculation that

$$\boldsymbol{e}_{123}\cdot\boldsymbol{e}_{123} = -\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}\mathbf{I} \tag{118.1}$$

Turning for purposes of comparison to $\mathcal{C}_2[\mathbb{g}]$, where all indices range on $\{1,2\}$ and

$$\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{kl} = \pm\boldsymbol{e}_{12}\cdot\boldsymbol{e}_{12} \quad \text{if it does not vanish}$$

Working from (109.3/117.2) we have

$$\begin{aligned} \boldsymbol{e}_{12}\cdot\boldsymbol{e}_{12} &= \boldsymbol{e}_{1212} - (g_{11}\boldsymbol{e}_{22} + g_{22}\boldsymbol{e}_{11}) + (g_{21}\boldsymbol{e}_{12} + g_{12}\boldsymbol{e}_{21}) - (g_{11}g_{22} - g_{12}g_{21})\mathbf{I} \\ &= -\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}\mathbf{I} \end{aligned} \tag{118.2}$$

Equations (118) generalize statements that in the Euclidean case (where both determinants become unity) are obvious:

$$\begin{aligned} \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\cdot\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 &= (-)^3\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_3 = -\mathbf{I} \\ \boldsymbol{e}_1\boldsymbol{e}_2\cdot\boldsymbol{e}_1\boldsymbol{e}_2 &= (-)^1\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2 = -\mathbf{I} \end{aligned}$$

Our problem now is to make digestable sense of the information we have worked so hard to obtain. To that end I have entered (105), (107), (109) and (111) into *Mathematica* as functional definitions, writing

$$\begin{aligned} \texttt{oneone[i\_, j\_]:=}\ & e_{i,j} + g_{i,j} \\ \texttt{onetwo[i\_, j\_, k\_]:=}\ & e_{i,j,k} - g_{i,k}e_j + g_{i,j}e_k \\ \texttt{twoone[j\_, k\_, i\_]:=}\ & e_{i,j,k} + g_{i,k}e_j - g_{i,j}e_k \end{aligned}$$

etc. Working first within $\mathcal{C}_2[\boldsymbol{g}]$, we write

$$\begin{aligned}\boldsymbol{c} &\equiv s\,\mathsf{I} + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + p\,\boldsymbol{e}_{12}\\ \boldsymbol{C} &\equiv S\,\mathsf{I} + V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + P\,\boldsymbol{e}_{12}\end{aligned}$$

and, making free use of such little facts as $\boldsymbol{e}_{2,1} = -\boldsymbol{e}_{1,2}$ and $\boldsymbol{e}_{1,1,2} = \mathbf{0}$, obtain a result that can be expressed

$$\begin{aligned}\boldsymbol{c}\cdot\boldsymbol{C} = \mathsf{I}&\big[sS+(g_{11}v^1V^1+g_{12}v^1V^2+g_{21}v^2V^1+g_{22}v^2V^2)-(g_{11}g_{22}-g_{12}g_{21})pP\big]\\ &+\boldsymbol{e}_1\big[sV^1+v^1S-(v^1g_{12}+v^2g_{22})P+p(V^1g_{12}+V^2g_{22})\big]\\ &+\boldsymbol{e}_2\big[sV^2+v^2S+(v^1g_{11}+v^2g_{21})P-p(V^1g_{11}+V^2g_{21})\big]\\ &+\boldsymbol{e}_{12}\big[sP+v^1V^2-v^2V^1+pS\big]\end{aligned}$$

Here we have recovered precisely the multiplication formula that was presented as (75.1) on page 18 (and in notationally compacted form as (75.2) on page 20)... which is gratifying... and, as will soon emerge, useful in a surprising connection.

Turning now at last to $\mathcal{C}_3[\boldsymbol{g}]$, we discover that everything hinges upon *how we elect to display* the Clifford numbers in question. Suppose, for example, we were to yield to the natural temptation to write

$$\begin{aligned}\boldsymbol{c} &\equiv s\,\mathsf{I} + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + v^3\boldsymbol{e}_3 + a_1\boldsymbol{e}_{23} + a_2\boldsymbol{e}_{31} + a_3\boldsymbol{e}_{12} + p\,\boldsymbol{e}_{123}\\ \boldsymbol{C} &\equiv S\,\mathsf{I} + V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + V^3\boldsymbol{e}_3 + A_1\boldsymbol{e}_{23} + A_2\boldsymbol{e}_{31} + A_3\boldsymbol{e}_{12} + P\,\boldsymbol{e}_{123}\end{aligned}$$

We would confront then a fairly formidable computational problem: the 4^2 terms that in $\mathcal{C}_3[\boldsymbol{g}]$ entered into the development of $\boldsymbol{c}\cdot\boldsymbol{C}$ have become now 8^2 terms, and some if those are fairly complicated.[28] There is, I claim, a better way, but to describe it I must back up a bit:

Look by way of orientation to the case $\mathcal{C}_n[\boldsymbol{g}_{\text{Euclidean}}]$, wherein

$$\boldsymbol{e}_{ij} \text{ becomes } \boldsymbol{e}_i\boldsymbol{e}_j = \begin{cases} -\boldsymbol{e}_j\boldsymbol{e}_i & : \quad i \neq j\\ \mathsf{I} & : \quad i = j\end{cases}$$

Look more particularly to a property of the element $\boldsymbol{f} \equiv \boldsymbol{e}_{1,2,\ldots,n} = \boldsymbol{e}_1\boldsymbol{e}_2\cdots\boldsymbol{e}_n$. Clearly

$$\boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_p}\boldsymbol{f} = [(-)^{n-1}]^p\boldsymbol{f}\,\boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_p} \;:\; i_1 < i_2 < \cdots < i_p \;\;\&\;\; p \leqslant n$$

$$= \begin{cases} (-)^p\boldsymbol{f}\,\boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_p} & : \quad n \text{ even}\\ \boldsymbol{f}\,\boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_p} & : \quad n \text{ odd}\end{cases}$$

—the implication being that <u>if n is odd then $\boldsymbol{f}$ commutes with everything</u>: $\boldsymbol{f}$ has joined I as an element of the "center" of $\mathcal{C}_{\text{odd}}[\boldsymbol{g}_{\text{Euclidean}}]$, the general element

[28] In what is for me the case $\mathcal{C}_{12}[\boldsymbol{g}]$ of ultimate interest those would have expanded to a total of $[2^{12}]^2 = 16,777,216$ *very* complicated terms!

of which can be written $x\mathbf{I}+y\boldsymbol{f}$. Easily, $\boldsymbol{f}^2 = -\mathbf{I}$, so the center of such a Clifford algebra provides an abstract copy of the field of *complex* numbers.

Those properties of $\mathcal{C}_{\text{odd}}[\boldsymbol{g}_{\text{Euclidean}}]$ can be obtained as specialized instances of some properties of $\mathcal{C}_{\text{odd}}[\boldsymbol{g}]$—properties that I presently prepared to discuss only as they become manifest within $\mathcal{C}_3[\boldsymbol{g}]$. Let

$$\boldsymbol{f} \equiv \boldsymbol{e}_{123}$$

and understand that in non-Euclidean cases $\boldsymbol{f}$ must *not* be confused with $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$. We are informed by *Mathematica* (who we supplied with all relevant information on the preceding page) that

$$\begin{aligned} \boldsymbol{f}\boldsymbol{e}_i - \boldsymbol{e}_i\boldsymbol{f} &= 2\,\boldsymbol{e}_{123i} \\ &= \mathbf{0} \text{ in all cases: } i \in \{1,2,3\} \end{aligned} \tag{119.1}$$

$$\begin{aligned} \boldsymbol{f}\boldsymbol{e}_{ij} - \boldsymbol{e}_{ij}\boldsymbol{f} &= 2(g_{i1}\boldsymbol{e}_{j23} - g_{j1}\boldsymbol{e}_{i23}) + 2(g_{i2}\boldsymbol{e}_{j31} - g_{j2}\boldsymbol{e}_{i31}) + 2(g_{i3}\boldsymbol{e}_{j12} - g_{j3}\boldsymbol{e}_{i12}) \\ &= \mathbf{0} \text{ in all cases: } i,j \in \{1,2,3\} \end{aligned} \tag{119.2}$$

and that

$$\boldsymbol{f}^2 = -g\,\mathbf{I} \quad \text{with} \quad g \equiv \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \tag{120}$$

Now a trick. Take the basic elements of $\mathcal{C}_3[\boldsymbol{g}]$ to be

$$\{\mathbf{I}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{12}, \boldsymbol{f}, \boldsymbol{f}\boldsymbol{e}_1, \boldsymbol{f}\boldsymbol{e}_2, \boldsymbol{f}\boldsymbol{e}_{12}\} \tag{121}$$

—noting that while <u>in the Euclidean case</u> $\mathcal{C}_3[\boldsymbol{g}_{\text{Euclidean}}]$

$$\left.\begin{aligned} \boldsymbol{f} &\equiv \boldsymbol{e}_{123} && \text{becomes } \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \\ \boldsymbol{f}\boldsymbol{e}_1 &= \boldsymbol{e}_{123}\boldsymbol{e}_1 && \text{becomes } \boldsymbol{e}_2\boldsymbol{e}_3 \\ \boldsymbol{f}\boldsymbol{e}_2 &= \boldsymbol{e}_{123}\boldsymbol{e}_2 && \text{becomes } -\boldsymbol{e}_1\boldsymbol{e}_3 = \boldsymbol{e}_3\boldsymbol{e}_1 \\ \boldsymbol{f}\boldsymbol{e}_{12} &= \boldsymbol{e}_{123}\boldsymbol{e}_{12} && \text{becomes } -\boldsymbol{e}_3 \end{aligned}\right\} \tag{122}$$

three of the last four members of the proposed basic element set are in the general case somewhat goofy (*not* the sort of thing we would have plucked from thin air): *Mathematica* supplies

$$\left.\begin{aligned} \boldsymbol{f}\boldsymbol{e}_1 &= g_{11}\boldsymbol{e}_{23} + g_{12}\boldsymbol{e}_{31} + g_{13}\boldsymbol{e}_{12} \\ \boldsymbol{f}\boldsymbol{e}_2 &= g_{21}\boldsymbol{e}_{23} + g_{22}\boldsymbol{e}_{31} + g_{23}\boldsymbol{e}_{12} \\ \boldsymbol{f}\boldsymbol{e}_{12} &= -(g_{12}g_{23} - g_{13}g_{22})\boldsymbol{e}_1 \\ &\quad\; -(g_{13}g_{21} - g_{11}g_{23})\boldsymbol{e}_2 \\ &\quad\; -(g_{11}g_{22} - g_{12}g_{21})\boldsymbol{e}_3 \end{aligned}\right\} \tag{123}$$

from which, it will be noted, we can recover the Euclidean statements (122) as specialized consequences.

The point of the trick—of the seemingly unnatural element-selection (120) —is that it permits us portray $\mathcal{C}_3[\boldsymbol{g}]$ as a kind of "complex extension" of $\mathcal{C}_2[\boldsymbol{g}]$

$$\mathcal{C}_3[\boldsymbol{g}] = \mathcal{C}_2[\boldsymbol{g}] + \boldsymbol{f}\mathcal{C}_2[\boldsymbol{g}] \tag{124}$$

and to extract the rule for multiplying within $\mathcal{C}_3[\boldsymbol{g}]$ from the much simpler rule (75.2) that describes multiplication within $\mathcal{C}_2[\boldsymbol{g}]$.[29] To see how that works, and where it leads, let $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ be arbitrary elements of $\mathcal{C}_2[\boldsymbol{g}]$, and from them form

$$\boldsymbol{c} \equiv \boldsymbol{a} + \boldsymbol{f}\boldsymbol{b} \quad \text{and} \quad \boldsymbol{C} \equiv \boldsymbol{A} + \boldsymbol{f}\boldsymbol{B} \;:\; \text{elements of } \mathcal{C}_3[\boldsymbol{g}] \tag{125}$$

Then, drawing extensively upon (119) and (120),

$$\begin{aligned} \boldsymbol{c}\cdot\boldsymbol{C} &= (\boldsymbol{aA} + \boldsymbol{f}^2\boldsymbol{bB}) + \boldsymbol{f}(\boldsymbol{aB} + \boldsymbol{bA}) \\ &= (\boldsymbol{aA} - \;\; g\,\boldsymbol{bB}) + \boldsymbol{f}(\boldsymbol{aB} + \boldsymbol{bA}) \end{aligned}$$

To assign specific meaning to the expression on the right we have only to work out four products within $\mathcal{C}_2[\boldsymbol{g}]$.

Some valuable conclusions are fairly immediate. Suppose, for example, we were to

$$\text{set} \quad \boldsymbol{C} \equiv \boldsymbol{a} - \boldsymbol{f}\boldsymbol{b} \tag{126.1}$$

We would then have

$$\boldsymbol{c}\cdot\boldsymbol{C} = (\boldsymbol{aa} + g\,\boldsymbol{bb}) - \boldsymbol{f}(\boldsymbol{ab} - \boldsymbol{ba}) \tag{126.2}$$

which, however, puts us within sight of a formula for $\boldsymbol{c}^{-1}$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ commute ($\boldsymbol{ab} - \boldsymbol{ba} = \boldsymbol{0}$), which is a severe restriction. To escape the force of this difficulty we note that the following elements of $\mathcal{C}_3[\boldsymbol{g}]$

$$\boldsymbol{P}_+ \equiv \tfrac{1}{2}(\mathbf{I} + ig^{-\frac{1}{2}}\boldsymbol{f}) \qquad \text{and} \qquad \boldsymbol{P}_- \equiv \tfrac{1}{2}(\mathbf{I} - ig^{-\frac{1}{2}}\boldsymbol{f}) \tag{127}$$

---

[29] It should be noticed that the $\boldsymbol{g}$ on the right side of (124) is 2×2, while the $\boldsymbol{g}$ on the left is 3×3. The metric components

$$\begin{pmatrix} & & g_{13} \\ & & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

are, however, not actually missing on the right: they are, according to (123), *sequestered in the definitions* of $\boldsymbol{f}\boldsymbol{e}_1$, $\boldsymbol{f}\boldsymbol{e}_2$ and $\boldsymbol{f}\boldsymbol{e}_{12}$.

mimic the properties of a *complete set of orthogonal projection operators*:[30]

$$\boldsymbol{P}_+ + \boldsymbol{P}_- = \mathbf{I} \tag{128.1}$$

$$\boldsymbol{P}_+ \cdot \boldsymbol{P}_- = \boldsymbol{P}_- \cdot \boldsymbol{P}_+ = \mathbf{0} \tag{128.2}$$

$$\boldsymbol{P}_+^2 = \boldsymbol{P}_+ \quad \text{and} \quad \boldsymbol{P}_-^2 = \boldsymbol{P}_- \tag{128.3}$$

Moreover, each *commutes with every element of* $\mathcal{C}_3[\boldsymbol{g}]$:

$$\boldsymbol{P}_\pm \boldsymbol{c} = \boldsymbol{c}\boldsymbol{P}_\pm \quad : \quad \text{all } \boldsymbol{c} \text{ in } \mathcal{C}_3[\boldsymbol{g}] \tag{128.4}$$

It follows that every such $\boldsymbol{c}$ can be presented as a *sum of orthogonal components*:

$$\boldsymbol{c} = \boldsymbol{c}_+ + \boldsymbol{c}_- \quad : \quad \boldsymbol{c}_+ \cdot \boldsymbol{c}_- = \mathbf{0} \quad \text{with} \quad \boldsymbol{c}_\pm \equiv \boldsymbol{P}_\pm \boldsymbol{c} \tag{129}$$

Fundamental to the inversion problem within $\mathcal{C}_3[\boldsymbol{g}]$ is the observation that so, in particular, can $\mathbf{I}$ be thus decomposed:

$$\mathbf{I} = \mathbf{I}_+ + \mathbf{I}_- \quad : \quad \mathbf{I}_+ \cdot \mathbf{I}_- = \mathbf{0} \quad \text{with} \quad \mathbf{I}_\pm \equiv \boldsymbol{P}_\pm \mathbf{I} \tag{130}$$

We are in position now to write

$$\boldsymbol{c} \cdot \boldsymbol{C} = (\boldsymbol{c}_+ + \boldsymbol{c}_-) \cdot (\boldsymbol{C}_+ + \boldsymbol{C}_-) = \boldsymbol{c}_+ \cdot \boldsymbol{C}_+ + \boldsymbol{c}_- \cdot \boldsymbol{C}_- \tag{131}$$

The inversion of $\boldsymbol{c}$ would be accomplished if we could arrange to have $\boldsymbol{c}_+ \cdot \boldsymbol{C}_+ = \mathbf{I}_+$ and $\boldsymbol{c}_- \cdot \boldsymbol{C}_- = \mathbf{I}_-$. Drawing *now* upon (this slight adjustment

$$\boldsymbol{c} \equiv \boldsymbol{a} + g^{-\frac{1}{2}} \boldsymbol{f} \boldsymbol{b} \tag{132}$$

of) the decomposition introduced on page 47 we find that[31]

$$\left.\begin{aligned} \boldsymbol{c}_+ &= (\boldsymbol{a} - i\boldsymbol{b})\mathbf{I}_+ \\ \boldsymbol{c}_- &= (\boldsymbol{a} + i\boldsymbol{b})\mathbf{I}_- \end{aligned}\right\} \tag{133}$$

---

[30] In one writes $\boldsymbol{P} \equiv \alpha(\mathbf{I} + \beta \boldsymbol{f})$ and requires $\boldsymbol{P}^2 = \boldsymbol{P}$ one finds that $\alpha = \frac{1}{2}$ and $\beta = \pm i g^{-\frac{1}{2}}$ are forced. P. K. Raševskiĭ—in "The theory of spinors," American Mathematical Society Translations, Series 2, Volume 6 (1957)—has provided an elaborate account of the theory of Clifford algebras in the Euclidean case. In the following discussion I enlarge upon material to be found in his §5.

[31] Use

$$\left.\begin{aligned} \boldsymbol{P}_+ \cdot g^{-\frac{1}{2}} \boldsymbol{f} &= -i\boldsymbol{P}_+ \\ \boldsymbol{P}_- \cdot g^{-\frac{1}{2}} \boldsymbol{f} &= +i\boldsymbol{P}_- \end{aligned}\right\} \tag{137}$$

which follow quickly from the projectivity statements

$$\boldsymbol{P}_+ \cdot \tfrac{1}{2}(\mathbf{I} + i g^{-\frac{1}{2}} \boldsymbol{f}) = \boldsymbol{P}_+ \quad \text{and} \quad \boldsymbol{P}_- \cdot \tfrac{1}{2}(\mathbf{I} - i g^{-\frac{1}{2}} \boldsymbol{f}) = \boldsymbol{P}_-$$

It should be noted also that $\boldsymbol{P}_+$ and $\mathbf{I}_+$ are different names for *the same thing* (ditto $\boldsymbol{P}_-$ and $\mathbf{I}_-$).

But $(\boldsymbol{a}-i\boldsymbol{b})$ and $(\boldsymbol{a}+i\boldsymbol{b})$ live in $\mathcal{C}_2[\boldsymbol{g}]$, where the inversion problem has already been solved. We are in position, therefore, to construct

$$\boldsymbol{c}^{-1} \equiv (\boldsymbol{a}-i\boldsymbol{b})^{-1}\,\mathbf{I}_+ + (\boldsymbol{a}+i\boldsymbol{b})^{-1}\,\mathbf{I}_- \tag{134}$$

and to observe that $\boldsymbol{c}\cdot\boldsymbol{c}^{-1} = \mathbf{I}_+ + \mathbf{I}_- = \mathbf{I}$.

I find it easy to resist any temptation to pursue the general-metric aspect of this discussion to its finer details.[32] The lesson, in general terms, is that

$$\mathcal{C}_3[\boldsymbol{g}] = \mathcal{C}_2'[\boldsymbol{g}] \oplus \mathcal{C}_2''[\boldsymbol{g}] \quad \text{with} \quad \mathcal{C}_2'[\boldsymbol{g}] \perp \mathcal{C}_2''[\boldsymbol{g}] \tag{135}$$

If matrices of the form $\begin{pmatrix}\bullet & \bullet\\ \bullet & \bullet\end{pmatrix}$ serve to represent elements of $\mathcal{C}_2[\boldsymbol{g}]$ then we expect to have

$$\begin{pmatrix}\bullet & \bullet & 0 & 0\\ \bullet & \bullet & 0 & 0\\ 0 & 0 & \bullet & \bullet\\ 0 & 0 & \bullet & \bullet\end{pmatrix} \tag{136}$$

in representation of $\mathcal{C}_3[\boldsymbol{g}]$. And we expect the transformation theory latent in $\mathcal{C}_3[\boldsymbol{g}]$ to be relatively uninteresting—to consist simply of duplex copies of the theory latent already in $\mathcal{C}_2[\boldsymbol{g}]$.

In the preceding discussion I surpressed fine details in order to expose most clearly the essential drift of the idea, but did so at cost: the discussion took place at such an abstract level that it is difficult to gain a vivid sense of what it was we actually accomplished. To remedy this defect I propose to revert now to the Euclidean metric, where everything is especially simple. Let

$$\left.\begin{aligned}\boldsymbol{a} &= a^0\mathbf{I} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_1\boldsymbol{e}_2\\ \boldsymbol{b} &= b^0\mathbf{I} + b^1\boldsymbol{e}_1 + b^2\boldsymbol{e}_2 + b^3\boldsymbol{e}_1\boldsymbol{e}_2\end{aligned}\right\} \tag{138}$$

and with the aid of

$$\boldsymbol{f} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \tag{139}$$

construct

$$\begin{aligned}\boldsymbol{c} &= \boldsymbol{a} + \boldsymbol{f}\boldsymbol{b}\\ &= (a^0\mathbf{I} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_1\boldsymbol{e}_2) + (b^0\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 + b^1\boldsymbol{e}_2\boldsymbol{e}_3 + b^2\boldsymbol{e}_3\boldsymbol{e}_1 - b^3\boldsymbol{e}_3)\end{aligned}$$

Use $\boldsymbol{P}_\pm \equiv \frac{1}{2}(\mathbf{I} \pm i\,\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3) \equiv \mathbf{I}_\pm$ to construct

$$\begin{aligned}\boldsymbol{c}_+ &= \boldsymbol{P}_\pm\,\boldsymbol{c}\\ &= \tfrac{1}{2}\big\{(a^0-ib^0)\mathbf{I} + (a^1-ib^1)\boldsymbol{e}_1 + (a^2-ib^2)\boldsymbol{e}_2 + (a^3-ib^3)\boldsymbol{e}_1\boldsymbol{e}_2\big\}\\ &\quad + i\,\tfrac{1}{2}\big\{(a^0-ib^0)\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 + (a^1-ib^1)\boldsymbol{e}_2\boldsymbol{e}_3 + (a^2-ib^2)\boldsymbol{e}_3\boldsymbol{e}_1 + (a^3-ib^3)\boldsymbol{e}_3\big\}\\ &= (\boldsymbol{a}-i\boldsymbol{b})\,\mathbf{I}_+ \qquad (140.1)\\ \boldsymbol{c}_- &= (\boldsymbol{a}+i\boldsymbol{b})\,\mathbf{I}_- \qquad (140.2)\end{aligned}$$

[32] Were we to do so we would, in particular, want to make clear where the "border elements[29] of the 3 × 3 metric matrix" have come finally to rest.

in terms of which we have

$$\begin{aligned}\boldsymbol{c} &= \boldsymbol{c}_+ + \boldsymbol{c}_- \\ &= (\boldsymbol{a} - i\boldsymbol{b})\,\mathsf{I}_+ + (\boldsymbol{a} + i\boldsymbol{b})\,\mathsf{I}_- \\ &= \boldsymbol{a}\,(\mathsf{I}_+ + \mathsf{I}_-) + \boldsymbol{b}\,(-i\,\mathsf{I}_+ + i\,\mathsf{I}_-) \\ &= \boldsymbol{a}\,(\mathsf{I}_+ + \mathsf{I}_-) + \boldsymbol{b}\,\boldsymbol{f}(\mathsf{I}_+ + \mathsf{I}_-) \\ &= \boldsymbol{a} + \boldsymbol{f}\boldsymbol{b}\end{aligned}$$

The claim is that $\boldsymbol{c}^{-1}$ can be described

$$\boldsymbol{c}^{-1} = (\boldsymbol{a} - i\boldsymbol{b})^{-1}\,\mathsf{I}_+ + (\boldsymbol{a} + i\boldsymbol{b})^{-1}\,\mathsf{I}_- \tag{141}$$

where—as was established already on page 5—

$$\begin{aligned}(\boldsymbol{a} + i\boldsymbol{b})^{-1} &= \frac{(a^0 + ib^0)\mathsf{I} - (a^1 + ib^1)\boldsymbol{e}_1 - (a^2 + ib^2)\boldsymbol{e}_2 - (a^3 + ib^3)\boldsymbol{e}_1\boldsymbol{e}_2}{(a^0 + ib^0)^2 - (a^1 + ib^1)^2 - (a^2 + ib^2)^2 + (a^3 + ib^3)^2} \\ (\boldsymbol{a} - i\boldsymbol{b})^{-1} &= \text{result of obvious adjustment: } i \to -i\end{aligned}$$

I propose now to develop a *matrix representation* of the algebra described on the preceding page, but to gain the advantage of expository efficiency must first digress to summarize the essential properties of the so-called **Kronecker product**.[33]

The "Kronecker product" (sometimes called the "direct product") of

- an $m \times n$ matrix $\mathbb{A}$ onto
- a $p \times q$ matrix $\mathbb{B}$

is the $mp \times nq$ matrix defined[34]

$$\mathbb{A} \otimes \mathbb{B} \equiv \|a_{ij}\,\mathbb{B}\| \tag{142}$$

Manipulation of expressions involving Kronecker products is accomplished by appeal to general statements such as the following:

$$k(\mathbb{A} \otimes \mathbb{B}) = (k\mathbb{A}) \otimes \mathbb{B} = \mathbb{A} \otimes (k\mathbb{B}) \tag{143.1}$$

$$\left.\begin{aligned}(\mathbb{A} + \mathbb{B}) \otimes \mathbb{C} &= \mathbb{A} \otimes \mathbb{C} + \mathbb{B} \otimes \mathbb{C} \\ \mathbb{A} \otimes (\mathbb{B} + \mathbb{C}) &= \mathbb{A} \otimes \mathbb{B} + \mathbb{A} \otimes \mathbb{C}\end{aligned}\right\} \tag{143.2}$$

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C} \equiv \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C} \tag{143.3}$$

---

[33] The following material was lifted directly from Chapter 1 page 24 of my ADVANCED QUANTUM TOPICS (2000).

[34] The alternative definition $\mathbb{A} \otimes \mathbb{B} \equiv \|\mathbb{A}\,b_{ij}\|$ gives rise to a "mirror image" of the standard theory. Good discussions can be found in E. P. Wigner, *Group Theory and its Application to the Quantum Theory of Atomic Spectra* (1959), Chapter 2; P. Lancaster, *Theory of Matrices* (1969), §8.2; Richard Bellman, *Introduction to Matrix Analysis* (2nd edition 1970), Chapter 12, §§5–13.

$$(\mathbb{A}\otimes\mathbb{B})^{\mathsf{T}} = \mathbb{A}^{\mathsf{T}}\otimes\mathbb{B}^{\mathsf{T}} \tag{143.4}$$

$$\mathrm{tr}(\mathbb{A}\otimes\mathbb{B}) = \mathrm{tr}\mathbb{A}\cdot\mathrm{tr}\mathbb{B} \tag{143.5}$$

—all of which are valid except when meaningless.[35] Less obviously (but often very usefully)

$$(\mathbb{A}\otimes\mathbb{B})(\mathbb{C}\otimes\mathbb{D}) = \mathbb{AC}\otimes\mathbb{BD} \quad \text{if} \quad \begin{cases}\mathbb{A}\text{ and }\mathbb{C}\text{ are }m\times m\\ \mathbb{B}\text{ and }\mathbb{D}\text{ are }n\times n\end{cases} \tag{143.6}$$

from which one can extract[36]

$$\mathbb{A}\otimes\mathbb{B} = (\mathbb{A}\otimes\mathbb{I}_n)(\mathbb{I}_m\otimes\mathbb{B}) \tag{143.7}$$

$$\det(\mathbb{A}\otimes\mathbb{B}) = (\det\mathbb{A})^n(\det\mathbb{B})^m \tag{143.8}$$

$$(\mathbb{A}\otimes\mathbb{B})^{-1} = \mathbb{A}^{-1}\otimes\mathbb{B}^{-1} \tag{143.9}$$

Here I have used $\mathbb{I}_m$ to designate the $m\times m$ identity matrix; when the dimension is obvious from the context I will, in the future, allow myself to omit the subscript. The identities (143) are proven in each case by direct computation, and their great power will soon become evident. *Mathematica* can be enlisted to perform computations in this area (and can, in particular, be used to demonstrate the accuracy of (143)), but the procedure is a little fussy. If $\mathbb{A}$ and $\mathbb{B}$ are presented as lists of lists then the command

```
Outer[Times, A, B] // MatrixForm
```

permits one to inspect the design of $\mathbb{A}\otimes\mathbb{B}$:

EXAMPLE: Construct

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} p & q\\ r & s\end{pmatrix}$$

and let the outputs be called `A` and `B`:

$$\texttt{A} = \{\{a,b\},\{c,d\}\} \quad\text{and}\quad \texttt{B} = \{\{p,q\},\{r,s\}\}$$

The `Outer` command then produces

$$\begin{pmatrix} \begin{pmatrix} ap & aq\\ ar & as\end{pmatrix} & \begin{pmatrix} bp & bq\\ br & bs\end{pmatrix}\\ \begin{pmatrix} cp & cq\\ cr & cs\end{pmatrix} & \begin{pmatrix} dp & dq\\ dr & ds\end{pmatrix}\end{pmatrix}$$

[35] Recall that one cannot add matrices unless they are co-dimensional, and does not speak of the trace of a matrix unless it is square.
[36] See Lancaster[32] for the detailed arguments.

> Which is informative. But the interior braces—which are not easy to remove by hand—cause that object to behave improperly when subjected to such basic matrix commands as `Det[ ]`, `Inverse[ ]`, `Transpose[ ]`.

I have devised a command that is free from that limitation—that yields output that *is* susceptible to routine matrix manipulation—but it is complicated:[37]

```
A_⊗B_:=
  Flatten[
    Table[Flatten[Table[Part[Outer[Times,A,B],i,j,k],
      {j, Dimensions[A][[2]]}]], {i, Dimensions[A][[1]]},
      {k, Dimensions[B][[1]]}],1]
```

It would be interesting to learn of a briefer command that serves equally well to create

$$\mathbb{A}\otimes\mathbb{B}=\begin{pmatrix} ap & aq & bp & bq\\ ar & as & br & bs\\ cp & cq & dp & dq\\ cr & cs & dr & ds\end{pmatrix}$$

To construct my matrix representation I pull from my intuitive hat the hunch that the projection numbers $\boldsymbol{P}_+$ and $\boldsymbol{P}_-$ might most naturally/usefully be represented by the complete pair of orthogonal projection matrices

$$\mathbb{P}_+\equiv\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\mathbb{I}_2$$
$$\mathbb{P}_-\equiv\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\mathbb{I}_2$$

which, while they do not commute with *every* 4×4 matrix, do commute with every matrix of the form (136). From the matrix representation

$$\mathbb{P}_+=\tfrac{1}{2}(\mathbb{I}+i\mathbb{F})\quad\text{and}\quad\mathbb{P}_-=\tfrac{1}{2}(\mathbb{I}-i\mathbb{F})$$

of (127) we are brought to the forced conclusion that

$$\mathbb{F}=-i(\mathbb{P}_+-\mathbb{P}_-)=\begin{pmatrix}-i&0&0&0\\0&-i&0&0\\0&0&+i&0\\0&0&0&+i\end{pmatrix}$$

---

[37] To create 〚, 〛 and ⊗ at the *Mathematica* keyboard type ESC[[ESC, ESC]]ESC and ESCc*ESC, respectively.

We will borrow our representations of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ from (45) on page 11:

$$\mathbb{E}_1 = \mathbb{I} \otimes \boldsymbol{e}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad : \quad \mathbb{I} \equiv \mathbb{I}_2$$

$$\mathbb{E}_2 = \mathbb{I} \otimes \boldsymbol{e}_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix}$$

—in which connection we note that $\mathbb{F}$, as developed above, can be described

$$\mathbb{F} = \boldsymbol{e}_2\boldsymbol{e}_1 \otimes \mathbb{I}$$

From $\boldsymbol{e}_3 = (\boldsymbol{e}_1\boldsymbol{e}_2)^{-1}\boldsymbol{f} = \boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{f}$ it then follows that necessarily

$$\begin{aligned} \mathbb{E}_3 = \mathbb{E}_2\mathbb{E}_1\mathbb{F} &= (\mathbb{I} \otimes \boldsymbol{e}_2)(\mathbb{I} \otimes \boldsymbol{e}_1)(\boldsymbol{e}_2\boldsymbol{e}_1 \otimes \mathbb{I}) \\ &= \boldsymbol{e}_1\boldsymbol{e}_2 \otimes \boldsymbol{e}_1\boldsymbol{e}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

It would at this point be very easy to demonstrate that the multiplicative properties of $\{\mathbb{I}, \mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_2\mathbb{E}_3, \mathbb{E}_3\mathbb{E}_1, \mathbb{E}_1\mathbb{E}_2, \mathbb{E}_1\mathbb{E}_2\mathbb{E}_3\}$ precisely mimic those of $\{\mathbf{I}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_3\boldsymbol{e}_1, \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\}$ . . . but I won't.

Now take

$$\begin{aligned} \mathbb{A} &\equiv a^0\mathbb{I} + a^1\mathbb{E}_1 + a^2\mathbb{E}_2 + a^3\mathbb{E}_1\mathbb{E}_2 \\ \mathbb{B} &\equiv b^0\mathbb{I} + b^1\mathbb{E}_1 + b^2\mathbb{E}_2 + b^3\mathbb{E}_1\mathbb{E}_2 \end{aligned}$$

and from them construct

$$\begin{aligned} \mathbb{C} &= \mathbb{A} + \mathbb{F}\,\mathbb{B} \\ &= a^0\mathbb{I} + a^1\mathbb{E}_1 + a^2\mathbb{E}_2 + b^3\mathbb{E}_3 + b^1\mathbb{E}_2\mathbb{E}_3 + b^2\mathbb{E}_3\mathbb{E}_1 + a^3\mathbb{E}_1\mathbb{E}_2 + b^0\mathbb{E}_1\mathbb{E}_2\mathbb{E}_3 \end{aligned}$$

Mathematica reports that $\mathbb{C}$ is of the form (136), and more specifically that

$$\mathbb{C} = \begin{pmatrix} \mathbb{C}_+ & \mathbb{O} \\ \mathbb{O} & \mathbb{C}_- \end{pmatrix}$$

with

$$\mathbb{C}_+ = \begin{pmatrix} (a^0 + ia^3) - i(b^0 + ib^3) & (a^1 - ia^2) - i(b^1 - ib^2) \\ (a^1 + ia^2) - i(b^1 + ib^2) & (a^0 - ia^3) - i(b^0 - ib^3) \end{pmatrix}$$

$$\mathbb{C}_- = \begin{pmatrix} (a^0 + ia^3) + i(b^0 + ib^3) & (a^1 - ia^2) + i(b^1 - ib^2) \\ (a^1 + ia^2) + i(b^1 + ib^2) & (a^0 - ia^3) + i(b^0 - ib^3) \end{pmatrix}$$

Clearly

$$\mathbb{C}'\cdot\mathbb{C}'' = \begin{pmatrix} \mathbb{C}'_+\cdot\mathbb{C}''_+ & \mathbb{O} \\ \mathbb{O} & \mathbb{C}'_-\cdot\mathbb{C}''_- \end{pmatrix}$$

The problem of multiplying two 4×4 matrices has been reduced to two instances of the problem of multiplying two 2×2 matrices. It is clear also that

$$\mathbb{C}^{-1} = \begin{pmatrix} \mathbb{C}_+^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{C}_-^{-1} \end{pmatrix}$$

and will exist if and only if

$$\det\mathbb{C} = \det(\mathbb{C}_+)\det(\mathbb{C}_-) \neq 0$$

These sweet results are made even sweeter by the observation that (see again equations (45) on page 9) the 2×2 matrices $\mathbb{C}_\pm$ can be developed

$$\begin{aligned}\mathbb{C}_+ &= (a^0 - ib^0)\mathbb{I} + (a^1 - ib^1)\mathbb{e}_1 + (a^2 - ib^2)\mathbb{e}_2 + (a^3 - ib^3)\mathbb{e}_1\mathbb{e}_2 \\ \mathbb{C}_- &= (a^0 + ib^0)\mathbb{I} + (a^1 + ib^1)\mathbb{e}_1 + (a^2 + ib^2)\mathbb{e}_2 + (a^3 + ib^3)\mathbb{e}_1\mathbb{e}_2\end{aligned}$$

so we have (*Mathematica* concurs)

$$\begin{aligned}\mathbb{C}_+^{-1} &= \frac{(a^0 - ib^0)\mathbb{I} - (a^1 - ib^1)\mathbb{e}_1 - (a^2 - ib^2)\mathbb{e}_2 - (a^3 - ib^3)\mathbb{e}_1\mathbb{e}_2}{\det\mathbb{C}_+} \\ \mathbb{C}_-^{-1} &= \frac{(a^0 + ib^0)\mathbb{I} - (a^1 + ib^1)\mathbb{e}_1 - (a^2 + ib^2)\mathbb{e}_2 - (a^3 + ib^3)\mathbb{e}_1\mathbb{e}_2}{\det\mathbb{C}_-}\end{aligned}$$

with

$$\det\mathbb{C}_\pm = (a^0 \mp ib^0)^2 - (a^1 \mp ib^1)^2 - (a^2 \mp ib^2)^2 + (a^3 \mp ib^3)^2$$

It follows that if $\boldsymbol{c} = a^0\mathbf{I} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + b^3\boldsymbol{e}_3 + b^1\boldsymbol{e}_2\boldsymbol{e}_3 + b^2\boldsymbol{e}_3\boldsymbol{e}_1 + a^3\boldsymbol{e}_1\boldsymbol{e}_2 + b^0\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ is real (in the sense that the a's and b's are real) then $\det\mathbb{C}$ is real, and given by

$$\det\mathbb{C} = |(a^0 \mp ib^0)^2 - (a^1 \mp ib^1)^2 - (a^2 \mp ib^2)^2 + (a^3 \mp ib^3)^2|^2$$

Notice also that

$$\begin{aligned}&\downarrow \\ &= (a^0a^0 - a^1a^1 - a^2a^2 + a^3a^3)^2 \text{ in the special case } \boldsymbol{b} = \mathbf{0}\end{aligned}$$

All reference to $\boldsymbol{e}_3$ has disappeared: we have recovered not the "modulus" previously encountered in the theory of $\mathcal{C}_2[\boldsymbol{g}_{\text{Euclidean}}]$,[38] but its square.

The real regular representation of $\mathcal{C}_3[\boldsymbol{g}_{\text{Euclidean}}]$ is 8-dimensional. The complex representation described above—extracted from the representation theory of $\mathcal{C}_2[\boldsymbol{g}_{\text{Euclidean}}]$—is 4-dimensional and it seems clear (though I have

[38] See again equations (28) & (47) on pages 5 & 10.

not proven) that its dimension is least-possible . I exists in many variants. An alternative representation would result if, for example, we sent

$$\begin{aligned}\mathbb{E}_1 &\mapsto \mathbb{E}_1' = \mathbb{E}_2\\ \mathbb{E}_2 &\mapsto \mathbb{E}_2' = \mathbb{E}_3\\ \mathbb{E}_3 &\mapsto \mathbb{E}_3' = \mathbb{E}_1\end{aligned}$$

which is, in effect, to proceed from a different $\mathcal{C}_2$ sub-algebra of $\mathcal{C}_3$: to assign to $\boldsymbol{e}_1$ the special role formerly assigned to $\boldsymbol{e}_3$. Or we could assume that $\mathbb{P}_\pm$ project onto some other/any orthogonal pair of planes in 4-space. Or we could subject the representation in hand to an arbitrary similarity transformation

$$\begin{aligned}\mathbb{E}_1 &\mapsto \mathbb{E}_1' = \mathbb{S}^{-1}\mathbb{E}_1\mathbb{S}\\ \mathbb{E}_2 &\mapsto \mathbb{E}_2' = \mathbb{S}^{-1}\mathbb{E}_2\mathbb{S}\\ \mathbb{E}_3 &\mapsto \mathbb{E}_3' = \mathbb{S}^{-1}\mathbb{E}_3\mathbb{S}\end{aligned}$$

We anticipate—though it could conceivably turn out to be otherwise—that *all* least-dimensional (or "irreducible") representations of $\mathcal{C}_3[\boldsymbol{g}_{\text{Euclidean}}]$ are interrelated in this manner. And that the representation theory of $\mathcal{C}_3[\boldsymbol{g}]$ is simply (or not so simply!) a fussed-up variant of the Euclidean theory.

6. Fourth order Clifford algebra with general metric. $\mathcal{C}_4[\boldsymbol{g}]$ is generated by objects $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ that satisfy relations of a sort

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2\,g_{ij}\,\mathbf{I} \tag{144}$$

that are characteristic of Clifford algebras in general, and that were first encountered in these pages at (72). Physicists, or course, have relativistic interest in the 4-metric

$$\boldsymbol{g}_{\text{Lorentz}} \equiv \|g_{\mu\nu}\| \equiv \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{145}$$

It was Dirac who first noticed the relevance of (144)—in that special instance—to relativistic quantum mechanics: he wrote[39]

$$\boldsymbol{\gamma}_\mu\boldsymbol{\gamma}_\nu + \boldsymbol{\gamma}_\nu\boldsymbol{\gamma}_\mu = 2\,g_{\mu\nu}\,\mathbf{I} \tag{146}$$
$$g_{\mu\nu} \text{ taken to be Lorentzian}$$

[39] Without reference to Clifford, whose name appears, so far as I am aware, nowhere in any of Dirac's published work.

. . . pulled a matrix representation

$$\Gamma_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Gamma_1 \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_2 \equiv \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma_3 \equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

out of thin air,[40] and was led directly to the celebrated *Dirac equation* (1928), which can (in this or any other irreducible representation) be considered to describe the motion of a 4-component wavefunction. The algebraic structure latent in (146) is known among physicists as the "Dirac algebra," about which a great deal has been written.

In 1960 I was motivated to consider what becomes of the Dirac algebra when the metric is allowed to become arbitrary: I was, in short, motivated (by considerations that I today find not very urgent!) to study $\mathcal{C}_4[\mathbb{g}]$. I will allow myself to borrow freely from that ancient material, which resides in what I will call my GENEVA NOTEBOOK. In view of the heavy demands which we found it necessary to make upon *Mathematica* in our effort to develop the theory of $\mathcal{C}_3[\mathbb{g}]$ I find it remarkable that—working only with pen and (large sheets of) paper—I was able to make any progress at all toward a theory of $\mathcal{C}_4[\mathbb{g}]$. But I was in fact able to carry that theory through to a kind of completion. I will be interested in reminding myself how that feat was accomplished.

The way the theory plays out depends critically upon what one takes to comprise the "basis set," in terms of which the elements $\boldsymbol{c}$ of $\mathcal{C}_4$ are to be developed as linear combinations. In the Euclidean case—or, more generally, if the metric is diagonal

$$\mathbb{g} = \begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{pmatrix}$$

[40] Not quite: the Pauli matrices (43) were already in circulation by 1928, and in terms of them we have the highly structured statements

$$\Gamma_0 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}$$

$$\Gamma_1 = \begin{pmatrix} \mathbb{O} & -\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & \mathbb{O} \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \mathbb{O} & -\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & \mathbb{O} \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} \mathbb{O} & -\boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_3 & \mathbb{O} \end{pmatrix}$$

My conventions here conform to those adopted in Appendix C of David Griffiths' *Introduction to Elementary Particles* (1987), and are fairly standard.

—it might make efficient good sense to work with (say)

$$\begin{aligned}
&\mathbf{I}\\
&\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_2,\ \boldsymbol{e}_4\\
&\boldsymbol{e}_1\boldsymbol{e}_2,\ \boldsymbol{e}_1\boldsymbol{e}_3,\ \boldsymbol{e}_1\boldsymbol{e}_4,\ \boldsymbol{e}_2\boldsymbol{e}_3,\ \boldsymbol{e}_2\boldsymbol{e}_4,\ \boldsymbol{e}_3\boldsymbol{e}_4\\
&\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4,\ \boldsymbol{e}_3\boldsymbol{e}_4\boldsymbol{e}_1,\ \boldsymbol{e}_4\boldsymbol{e}_1\boldsymbol{e}_2,\ \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\\
&\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4
\end{aligned}$$

But if the metric has non-zero off-diagonal elements then permuting the elements of such products does not simply introduce occasional minus signs: permutation brings additive shifts into play, as in $\boldsymbol{e}_2\boldsymbol{e}_1 = -\boldsymbol{e}_1\boldsymbol{e}_2 + g_{12}\mathbf{I}$. It was to blunt the force of this circumstance that at (99) we adopted the strategy of antisymmetrized averaging:

- In place of $\boldsymbol{e}_1\boldsymbol{e}_2$ adopt (compare (73) on page 18)
$$\boldsymbol{e}_{12} \equiv \frac{\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1}{2!} = \boldsymbol{e}_1\boldsymbol{e}_2 - g_{12}\mathbf{I}$$
- In place of $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ adopt
$$\boldsymbol{e}_{123} \equiv \frac{\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_3\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1}{3!}$$
- In place of $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4$ adopt
$$\boldsymbol{e}_{1234} = \tfrac{1}{4!}\varepsilon^{ijkl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$$

We are by this strategy led to a set of basis elements that organizes itself into "binomial piles:" we have

$$\begin{aligned}
\tbinom{4}{0} &= 1 \text{ term\ \ of type } \mathbf{I}\\
\tbinom{4}{1} &= 4 \text{ terms of type } \boldsymbol{e}_i\\
\tbinom{4}{2} &= 6 \text{ terms of type } \boldsymbol{e}_{ij}\\
\tbinom{4}{3} &= 4 \text{ terms of type } \boldsymbol{e}_{ijk}\\
\tbinom{4}{1} &= 1 \text{ term\ \ of type } \boldsymbol{e}_{ijkl}, \text{ call it } \boldsymbol{f} = \boldsymbol{e}_{1234}
\end{aligned}$$

In Dirac algebra (relativistic quantum applications) the $\boldsymbol{e}_i$'s are usually denoted $\boldsymbol{\gamma}_0,\ \boldsymbol{\gamma}_1,\ \boldsymbol{\gamma}_2,\ \boldsymbol{\gamma}_3$. We established at (108) that

$$\begin{aligned}
\boldsymbol{e}_{ijkl} = \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l &- (g_{ij}\boldsymbol{e}_{kl} + g_{kl}\boldsymbol{e}_{ij}) + (g_{ik}\boldsymbol{e}_{jl} + g_{jl}\boldsymbol{e}_{ik}) - (g_{il}\boldsymbol{e}_{jk} + g_{jk}\boldsymbol{e}_{il})\\
&- (g_{ij}g_{kl} - g_{ik}g_{jl} + g_{il}g_{jk})\mathbf{I}
\end{aligned}$$

where $i,\ j,\ k,\ l$ are necessarily distinct (a permutation of 1,2,3,4 in $\mathcal{C}_4$). It follows in particular that

$$\begin{aligned}
\boldsymbol{e}_{1234} &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 - (g_{12}\boldsymbol{e}_{34} + g_{34}\boldsymbol{e}_{12}) + (g_{13}\boldsymbol{e}_{24} + g_{24}\boldsymbol{e}_{13}) - (g_{14}\boldsymbol{e}_{23} + g_{23}\boldsymbol{e}_{14})\\
&\qquad - (g_{12}g_{34} - g_{13}g_{24} + g_{14}g_{23})\mathbf{I}\\
&\downarrow\\
&= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 \quad \text{if and only if the metric is diagonal}
\end{aligned}$$

In Dirac algebra—where the metric *is* diagonal—

$$\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \quad \text{is usually called } \boldsymbol{\gamma}_5$$

and is represented by

$$\mathbb{\Gamma}_0\mathbb{\Gamma}_1\mathbb{\Gamma}_2\mathbb{\Gamma}_3 \equiv \mathbb{\Gamma}_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

It is sometimes useful to notice that

$$\boldsymbol{e}_{1234} = \tfrac{1}{4!}\varepsilon^{ijkl}\boldsymbol{e}_i\boldsymbol{e}_{jkl} = \tfrac{1}{4!}\varepsilon^{jkli}\boldsymbol{e}_{jkl}\boldsymbol{e}_i = \tfrac{1}{4!}\varepsilon^{ijkl}\boldsymbol{e}_{ij}\boldsymbol{e}_{kl}$$

The point, if not made obvious by a moment's thought, can be established by *Mathematica*-assisted computation: construct

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\texttt{Signature[\{i,j,k,l\}]}\bigstar$$

where ★ refers serially to the expressions that appear on the right sides of equations (109), page 34.

Which brings me to the tricky case $\boldsymbol{e}_{ijk}$. These elements are of four types: $\boldsymbol{e}_{234}$, $\boldsymbol{e}_{341}$, $\boldsymbol{e}_{412}$ and $\boldsymbol{e}_{123}$. We would eliminate indicial clutter if we agreed to label each by the "missing index," as was suggested already on page 30. To that end we might write

$$\left.\begin{aligned} \boldsymbol{f}^{\,1} &\equiv -\tfrac{1}{3!}\varepsilon^{1ijk}\boldsymbol{e}_{ijk} \\ \boldsymbol{f}^{\,2} &\equiv -\tfrac{1}{3!}\varepsilon^{2ijk}\boldsymbol{e}_{ijk} \\ \boldsymbol{f}^{\,3} &\equiv -\tfrac{1}{3!}\varepsilon^{3ijk}\boldsymbol{e}_{ijk} \\ \boldsymbol{f}^{\,4} &\equiv -\tfrac{1}{3!}\varepsilon^{4ijk}\boldsymbol{e}_{ijk} \end{aligned}\right\} \tag{147.1}$$

—the curious minus signs will be motivated in a moment—and from those objects construct

$$\left.\begin{aligned} \boldsymbol{f}_1 &\equiv g_{1m}\boldsymbol{f}^{\,m} \\ \boldsymbol{f}_2 &\equiv g_{2m}\boldsymbol{f}^{\,m} \\ \boldsymbol{f}_3 &\equiv g_{3m}\boldsymbol{f}^{\,m} \\ \boldsymbol{f}_4 &\equiv g_{4m}\boldsymbol{f}^{\,m} \end{aligned}\right\} \tag{147.2}$$

In the diagonal case we would then have

$$\begin{aligned} \boldsymbol{f}_1 &= -g_{11}\boldsymbol{e}_{234} = -g_{11}\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 \\ \boldsymbol{f}_2 &= +g_{22}\boldsymbol{e}_{341} = +g_{22}\boldsymbol{e}_3\boldsymbol{e}_4\boldsymbol{e}_1 \\ \boldsymbol{f}_3 &= -g_{33}\boldsymbol{e}_{412} = -g_{33}\boldsymbol{e}_4\boldsymbol{e}_1\boldsymbol{e}_2 \\ \boldsymbol{f}_4 &= +g_{44}\boldsymbol{e}_{123} = +g_{44}\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \end{aligned}$$

In the GENEVA NOTEBOOK I chose, on the other hand, to introduce elements $\boldsymbol{\lambda}_i$ by the rule

$$\boldsymbol{\lambda}_i \equiv \boldsymbol{f}\boldsymbol{e}_i \tag{148}$$

In diagonal cases this is readily seen[41] to amount to a mere change of notation

$$\boldsymbol{\lambda}_i = \boldsymbol{f}_i \tag{149}$$

To see what happens in non-diagonal cases we look to (111.4) on page 40, which supplies

$$\boldsymbol{\lambda}_i = \boldsymbol{e}_{1234}\boldsymbol{e}_i = -(g_{i1}\boldsymbol{e}_{234} - g_{i2}\boldsymbol{e}_{134} + g_{i3}\boldsymbol{e}_{124} - g_{i4}\boldsymbol{e}_{123})$$

whence

$$\boldsymbol{\lambda}^i = g^{im}\boldsymbol{\lambda}_m = \underbrace{-(\delta^i{}_1\boldsymbol{e}_{234} - \delta^i{}_2\boldsymbol{e}_{134} + \delta^i{}_3\boldsymbol{e}_{124} - \delta^i{}_4\boldsymbol{e}_{123})}_{\text{precisely the object constructed on the right side of (147.1)}}$$

The implication is that (149) holds *universally*—for *all* metrics. We have accomplished two things: we have secured contact with the conventions adopted in the GENEVA NOTEBOOK, and we have added to the variety of ways in which the basic elements of $\mathcal{C}_4[\boldsymbol{g}]$ can be described.

The ***G***eneral element of $\mathcal{C}_4[\boldsymbol{g}]$ will be denoted

$$\boldsymbol{G} = S\,\mathbf{I} + V^i\boldsymbol{e}_i + \tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij} + A^j\boldsymbol{f}_j + P\boldsymbol{f}$$

where (as was anticipated already on page 20, and for reasons not yet explained)

- S is intended to suggest "Scaler"
- V is intended to suggest "Vector"
- T is intended to suggest "antisymmetric Tensor"
- A is intended to suggest "Axial vector" (or "pseudo-vector")
- P is intended to suggest "Pseudo-scaler"

Let a second element $\boldsymbol{H}$ be constructed similarly

$$\boldsymbol{H} = s\,\mathbf{I} + v^i\boldsymbol{e}_i + \tfrac{1}{2}t^{ij}\boldsymbol{e}_{ij} + a^j\boldsymbol{f}_j + p\,\boldsymbol{f}$$

---

[41] A typical calculation runs

$$\boldsymbol{\lambda}_1 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4\boldsymbol{e}_1 = (-)^3\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 = -g_{11}\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 = \boldsymbol{f}_1$$

The point of the "curious minus signs" has now become clear: they serve to establish precise agreement with the conventions adopted in the GENEVA NOTEBOOK.

Our ability to develop the algebraic theory[42] of $\mathcal{C}_4[\boldsymbol{g}]$ hinges on our ability to describe the product $\boldsymbol{G}\cdot\boldsymbol{H}$. We have already in hand much of the requisite information...for recall:[43]

- we obtained at (105) the canonical development of $\boldsymbol{e}_i\cdot\boldsymbol{e}_j$
- from the canonical development (106) of $\boldsymbol{p}_{ijk}$ we extracted descriptions of $\boldsymbol{e}_i\cdot\boldsymbol{e}_{jk}$ and $\boldsymbol{e}_{jk}\cdot\boldsymbol{e}_i$
- from the canonical development (108) of $\boldsymbol{p}_{ijkl}$ we extracted descriptions of $\boldsymbol{e}_i\cdot\boldsymbol{e}_{jkl}$, $\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{kl}$ and $\boldsymbol{e}_{jkl}\cdot\boldsymbol{e}_i$
- from the canonical development (110) of $\boldsymbol{p}_{ijklm}$ we extracted descriptions of $\boldsymbol{e}_i\cdot\boldsymbol{e}_{jklm}$, $\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{klm}$, $\boldsymbol{e}_{klm}\cdot\boldsymbol{e}_{ij}$ and $\boldsymbol{e}_{jklm}\cdot\boldsymbol{e}_i$
- from the canonical development (113) of $\boldsymbol{p}_{ijklmn}$ we extracted descriptions of $\boldsymbol{e}_{ij}\cdot\boldsymbol{e}_{klmn}$, $\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmn}$ and $\boldsymbol{e}_{klmn}\cdot\boldsymbol{e}_{ij}$

This information places us in position to describe products $\boldsymbol{G}\cdot\boldsymbol{H}$ in cases where $P=p=0$. What we still lack are descriptions of $\boldsymbol{f}_i\cdot\boldsymbol{f}$, $\boldsymbol{f}\cdot\boldsymbol{f}_i$ and $\boldsymbol{f}\cdot\boldsymbol{f}$. To construct those we might use our established procedures to

- ◦ produce the canonical development of $\boldsymbol{p}_{ijklmnr}$ and extract descriptions of $\boldsymbol{e}_{ijk}\cdot\boldsymbol{e}_{lmnr}$ and $\boldsymbol{e}_{lmnr}\cdot\boldsymbol{e}_{ijk}$
- ◦ produce the canonical development of $\boldsymbol{p}_{ijklmnrs}$ and extract a description of $\boldsymbol{e}_{ijkl}\cdot\boldsymbol{e}_{mnrs}$.

Such an approach would, however, be enormously laborious (so fast does $n!$ grow), even with the assistance of *Mathematica*. I will refrain from attempting to pursue this route until I have learned how to get *Mathematica* to do *all* the work. How, then, to proceed? My plan is (i) to review how far we can go toward the description of $\boldsymbol{G}\cdot\boldsymbol{H}$ on the basis of what we already know, then (ii) to bring into play the quite different methods employed in the GENEVA NOTEBOOK and (iii) check for consistency in the region where the two methods overlap.

For computational purposes we abandon the $\boldsymbol{f}$-notation, writing

$$\begin{aligned}\boldsymbol{G} &= S\,\mathbf{I} + V^i\boldsymbol{e}_i + \tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij} - \tfrac{1}{3!}A_m\varepsilon^{mijk}\boldsymbol{e}_{ijk} + P\tfrac{1}{4!}\varepsilon^{ijkl}\boldsymbol{e}_{ijkl}\\ \boldsymbol{H} &= s\,\mathbf{I} + v^i\boldsymbol{e}_i + \tfrac{1}{2}t^{ij}\boldsymbol{e}_{ij} - \tfrac{1}{3!}a_m\varepsilon^{mijk}\boldsymbol{e}_{ijk} + p\,\tfrac{1}{4!}\varepsilon^{ijkl}\boldsymbol{e}_{ijkl}\end{aligned}$$

but will revert to $\boldsymbol{f}$-notation when stating our final results. The products $S\,\mathbf{I}\cdot\boldsymbol{H}$ and $\boldsymbol{G}\cdot s\,\mathbf{I}$ are trivial, and contribute to $\boldsymbol{G}\cdot\boldsymbol{H}$ the following population of terms:

$$Ss\,\mathbf{I} + (Sv^i + sV^i)\boldsymbol{e}_i + \tfrac{1}{2}(St^{ij} + sT^{ij})\boldsymbol{e}_{ij} + (Sa^j + sA^j)\boldsymbol{f}_j + (Sp + sP)\boldsymbol{f}$$

From (105) we obtain

$$V^iv^j\boldsymbol{e}_i\boldsymbol{e}_j = (V_mv^m)\,\mathbf{I} + V^iv^j\boldsymbol{e}_{ij}$$

[42] As distinguished from (say) the irreducible representation theory, which is in many respects a separate problem.

[43] In the following remarks I revert to the notation

$$\boldsymbol{p}_{ijk\ldots n} \equiv \boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\cdots\boldsymbol{e}_n$$

where (as will henceforth be our casual practice) we have used g_{ij} to manipulate indices, writing $V_m \equiv g_{mn}V^n$. Resolving $V^i v^j$ into its symmetric/antisymmetric parts

$$V^i v^j = \tfrac{1}{2}(V^i v^j + V^j v^i) + \tfrac{1}{2}(V^i v^j - V^j v^i)$$

we note that—because $\boldsymbol{e}_{ij}$ is itself antisymmetric—only the antisymmetric part of $V^i v^j$ survives the summation process: we therefore have

$$V^i v^j \boldsymbol{e}_i \boldsymbol{e}_j = (V_m v^m)\mathbf{I} + \tfrac{1}{2}(V^i v^j - V^j v^i)\boldsymbol{e}_{ij} \tag{150.1}$$

Drawing next upon (107) we obtain

$$\tfrac{1}{2}(V^i\, t^{jk}\boldsymbol{e}_i\boldsymbol{e}_{jk} + v^i T^{jk}\boldsymbol{e}_{jk}\boldsymbol{e}_i) = (V_m t^{mi} - v_m T^{mi})\boldsymbol{e}_i + \tfrac{1}{2}(V^k t^{mn} + v^k T^{mn})\boldsymbol{e}_{kmn}$$

But from $\boldsymbol{f}^j = -\frac{1}{3!}\varepsilon^{jpqr}\boldsymbol{e}_{pqr}$ (see again (147.1) on page 59) it follows[44] that

$$\varepsilon_{jkmn}\boldsymbol{f}^j = -\tfrac{1}{3!}\varepsilon_{jkmn}\varepsilon^{jpqr}\boldsymbol{e}_{pqr} = -g\boldsymbol{e}_{kmn}$$

so we have

$$\begin{aligned}&\tfrac{1}{2}(V^i\, t^{jk}\boldsymbol{e}_i\boldsymbol{e}_{jk} + v^i T^{jk}\boldsymbol{e}_{jk}\boldsymbol{e}_i)\\ &\qquad = (V_m t^{mi} - v_m T^{mi})\boldsymbol{e}_i - \tfrac{1}{2g}\varepsilon^{jlmn}(V_l t_{mn} + v_l T_{mn})\boldsymbol{f}_j\end{aligned} \tag{150.2}$$

These implications of (109) and (111) are relatively straightforward:[45]

$$\begin{aligned}&-\tfrac{1}{3!}(V^i a_m \boldsymbol{e}_i\boldsymbol{e}_{jkl} + v^i A_m \boldsymbol{e}_{jkl}\boldsymbol{e}_i)\varepsilon^{mjkl}\\ &\qquad = \tfrac{1}{2}(V_m a_n - v_m A_n)\varepsilon^{mnij}\boldsymbol{e}_{ij} - \tfrac{1}{3!}(V^i a_m - v^i A_m)\varepsilon^{mjkl}\boldsymbol{e}_{ijkl}\\ &\qquad = \tfrac{1}{2}(V_m a_n - v_m A_n)\varepsilon^{mnij}\boldsymbol{e}_{ij} - (V^m a_m - v^m A_m)\boldsymbol{f}\end{aligned} \tag{150.3}$$

$$\begin{aligned}\tfrac{1}{4!}(V^i p\boldsymbol{e}_i\boldsymbol{e}_{jklm} + v^i P\boldsymbol{e}_{jklm}\boldsymbol{e}_i)\varepsilon^{jklm} &= \tfrac{1}{3!}(V_j p - v_j P)\varepsilon^{jklm}\boldsymbol{e}_{klm}\\ &= -(V^j p - v^j P)\boldsymbol{f}_j\end{aligned} \tag{150.4}$$

In consequence again of (109) we find

$$\tfrac{1}{4}T^{ij}t^{kl}\boldsymbol{e}_{ij}\boldsymbol{e}_{kl} = \tfrac{1}{4}T^{ij}t^{kl}\boldsymbol{e}_{ijkl} - T^i{}_m t^{jm}\boldsymbol{e}_{ij} - \tfrac{1}{2}T_{mn}t^{mn}\,\mathbf{I}$$

But $\boldsymbol{e}_{ijkl} = \frac{1}{g}\varepsilon_{ijkl}\boldsymbol{f}$ so (dismissing the irrelevant symmetric part of $T^i{}_m t^{jm}$) we have

$$\begin{aligned}= -\tfrac{1}{2}T_{mn}t^{mn}\,\mathbf{I} - \tfrac{1}{2}(T^i{}_m t^{jm} - T^j{}_m t^{im})\boldsymbol{e}_{ij}&\\ + \tfrac{1}{4g}\varepsilon^{klmn}T_{kl}t_{mn}\boldsymbol{f}&\end{aligned} \tag{150.5}$$

---

[44] I draw here upon properties of the Levi-Civita symbols and of the closely related "generalized Kronecker symbols" that are developed on pages 8 & 9 of "Electrodynamical applications of the exterior calculus" (1996).

[45] The only tricky point: because only four values are available to the indices, the expressions $\varepsilon^{mjkl}\boldsymbol{e}_{ijkl}$ vanish unless $m = i$ (and each of the non-vanishing expressions comes in 3! flavors).

Revisiting (111) we find

$$
\begin{aligned}
&-\tfrac{1}{3!2}(T^{ij}a_n\boldsymbol{e}_{ij}\boldsymbol{e}_{klm}+t^{ij}A_n\boldsymbol{e}_{klm}\boldsymbol{e}_{ij})\varepsilon^{nklm}\\
&\qquad=\ \tfrac{1}{3!2}(T^{ij}a_n-t^{ij}A_n)\,g_{ik}\,\varepsilon^{nklm}\,\boldsymbol{e}_{jlm}+5\text{ similar terms}\\
&\qquad\quad-\tfrac{1}{3!2}\varepsilon^{nklm}(T^{ij}a_n-t^{ij}A_n)g_{il}g_{jm}\boldsymbol{e}_k+5\text{ similar terms}
\end{aligned}
$$

We write $-\frac{1}{g}\varepsilon_{pjlm}\boldsymbol{f}^p$ in place of $\boldsymbol{e}_{jlm}$, draw upon the identity[44]

$$
\tfrac{1}{g}\varepsilon^{nklm}\varepsilon_{pjlm}=2\delta^{nk}{}_{pj}\equiv 2(\delta^n{}_p\delta^k{}_j-\delta^n{}_j\delta^k{}_p)
$$

and—working very carefully on a large piece of paper—notice that all "trace terms" (terms proportional to the $T^m{}_m$) cancel: we are led at last to a result that can be written

$$
=-\tfrac{1}{2}\varepsilon^{ilmn}(A_lt_{mn}+a_lT_{mn})\boldsymbol{e}_i+(A_mt^{mj}-a_mT^{mj})\boldsymbol{f}_j \tag{150.6}
$$

Drawing finally upon (115.1) we sharpen our pencils, take another large piece of paper and—after much consolidation—obtain

$$
\begin{aligned}
\tfrac{1}{3!3!}A_pa_q\varepsilon^{pijk}\varepsilon^{qlmn}\boldsymbol{e}_{ijk}\boldsymbol{e}_{lmn}=\tfrac{1}{36}A_pa_q\Big\{&9\varepsilon^{ipjk}\varepsilon_i{}^{qmn}\boldsymbol{e}_{jkmn}\\
&-18\varepsilon^{pimn}\varepsilon^{qj}{}_{mn}\boldsymbol{e}_{ij}-6\varepsilon^{pijk}\varepsilon^q{}_{ijk}\,\mathsf{I}\Big\}
\end{aligned}
$$

But[44]

$$
\varepsilon^{ipjk}\varepsilon_i{}^{qmn}\boldsymbol{e}_{jkmn}=g\begin{vmatrix}g^{pq}&g^{pm}&g^{pn}\\g^{jq}&g^{jm}&g^{jn}\\g^{kq}&g^{km}&g^{kn}\end{vmatrix}\boldsymbol{e}_{jkmn}=\mathbf{0}
$$

for the same reason that $g^{ij}\boldsymbol{e}_{ij\bullet\bullet}$ vanishes (antisymmetry kills symmetry). Drawing similarly upon

$$
\varepsilon^{pimn}\varepsilon^{qj}{}_{mn}=2!\,g\begin{vmatrix}g^{pq}&g^{pj}\\g^{iq}&g^{ij}\end{vmatrix}
$$

and

$$
\varepsilon^{pijk}\varepsilon^q{}_{ijk}=3!\,g\,g^{pq}
$$

we are led straightforwardly to

$$
\tfrac{1}{3!3!}A_pa_q\varepsilon^{pijk}\varepsilon^{qlmn}\boldsymbol{e}_{ijk}\boldsymbol{e}_{lmn}=-gA_na^n\,\mathsf{I}-\tfrac{1}{2}g(A^ia^j-A^ja^i)\boldsymbol{e}_{ij} \tag{150.7}
$$

From equations (150) it now follows that if

$$
\left.\begin{aligned}
\boldsymbol{G}&=S\,\mathsf{I}+V^i\boldsymbol{e}_i+\tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij}+A^j\boldsymbol{f}_j+P\boldsymbol{f}\\
\boldsymbol{H}&=\ s\,\mathsf{I}+\ v^i\boldsymbol{e}_i+\tfrac{1}{2}\,t^{ij}\boldsymbol{e}_{ij}\ +\ a^j\boldsymbol{f}_j+p\,\boldsymbol{f}
\end{aligned}\right\} \tag{151}
$$

then

$$
\begin{aligned}
\boldsymbol{G}\cdot\boldsymbol{H} = \;&\mathsf{I}\Big\{Ss + V_m v^m - \tfrac{1}{2}T_{mn}t^{mn} - gA_n a^n\Big\}\\
&+\boldsymbol{e}_i\Big\{(Sv^i + sV^i) + (V_m t^{mi} - v_m T^{mi})\\
&\qquad\qquad - \tfrac{1}{2}\varepsilon^{ilmn}(A_l t_{mn} + a_l T_{mn})\Big\}\\
&+\tfrac{1}{2}\boldsymbol{e}_{ij}\Big\{(St^{ij} + sT^{ij}) + (V^i v^j - V^j v^i)\\
&\qquad\qquad + \tfrac{1}{2}\varepsilon^{ijmn}(V_m a_n - V_n a_m)\\
&\qquad\qquad - (T^i{}_m t^{jm} - T^j{}_m t^{im})\\
&\qquad\qquad - \tfrac{1}{2}\varepsilon^{ijmn}(A_m v_n - A_n v_m)\\
&\qquad\qquad - g(A^i a^j - A^j a^i)\\
&\qquad\qquad - \tfrac{1}{2}\varepsilon^{ijmn}(Pt_{mn} + pT_{mn})\Big\}\\
&+\boldsymbol{f}_j\Big\{(Sa^j + sA^j) - \tfrac{1}{2g}\varepsilon^{jlmn}(V_l t_{mn} + v_l T_{mn})\\
&\qquad\qquad + (A_m t^{mj} - a_m T^{mj}) + (Pv^j - pV^j)\Big\}\\
&+\boldsymbol{f}\Big\{Sp - V_m a^m + \tfrac{1}{4g}\varepsilon^{klmn}T_{kl}t_{mn} + A_n v^n + Ps\Big\}\\
&+A^j p\,\boldsymbol{f}_j\boldsymbol{f} + Pa^j\,\boldsymbol{f}\boldsymbol{f}^j + Pp\,\boldsymbol{f}\boldsymbol{f}
\end{aligned}
$$

which is—so far as it goes (we are not yet in position to evaluate the red terms)—in precise agreement with the result reported on page 115 of the GENEVA NOTEBOOK.[46]

The method used to obtain the preceding (still fragmentary) result may have some claim to conceptual elegance (it is, in any event, conceptually straightforward), but is —at every turn—computationally quite burdensome. The method employed in the GENEVA NOTEBOOK is, on the other hand, quite inelegant, but is computationally so relatively efficient that—working only with pen and (large sheets of) paper I was able in 1960 to carry the $\boldsymbol{G}\cdot\boldsymbol{H}$ problem all the way to completion. Here—working with our present set of notational conventions—I undertake to construct a sketch that "Geneva method," which I will use to obtain descriptions of the missing red terms. The basic plan of attack is familiar already from page 26, and might be symbolized

$$
\begin{pmatrix} g_1 & 0 & 0 & 0\\ 0 & g_2 & 0 & 0\\ 0 & 0 & g_3 & 0\\ 0 & 0 & 0 & g_4 \end{pmatrix}
\longmapsto
\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}
$$

[46] See also page 11 of "Aspects of the theory of Clifford algebras," which are the notes for a seminar presented 27 March 1968 and can be found in COLLECTED SEMINARS 1963–1970.

We assume initially that the metric is diagonal, which is to say: that the generators satisfy

$$\boldsymbol{e}_i'\boldsymbol{e}_j' + \boldsymbol{e}_j'\boldsymbol{e}_i' = 2g_{ij}'\,\mathbf{I} \quad : \quad g_{ij}' \equiv \begin{cases} g_i & : \quad i = j \\ 0 & : \quad i \neq j \end{cases} \tag{152}$$

Taking the elements to be a familiar notational refinement (page 59) of those listed on page 58

$$\begin{aligned}
&\mathbf{I}\\
&\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3', \boldsymbol{e}_4'\\
&\boldsymbol{e}_{12}', \boldsymbol{e}_{13}', \boldsymbol{e}_{14}', \boldsymbol{e}_{23}', \boldsymbol{e}_{24}', \boldsymbol{e}_{34}' \quad : \quad \boldsymbol{e}_{ij}' \equiv \boldsymbol{e}_i'\boldsymbol{e}_j' = \begin{cases} g_i\mathbf{I} & : \quad i = j \\ -\boldsymbol{e}_j'\boldsymbol{e}_i' & : \quad i \neq j \end{cases}\\
&\boldsymbol{f}' \equiv \boldsymbol{e}_1'\boldsymbol{e}_2'\boldsymbol{e}_3'\boldsymbol{e}_4'\\
&\boldsymbol{f}_i' \equiv \boldsymbol{f}'\boldsymbol{e}_i' = \begin{cases} -g_1\boldsymbol{e}_2'\boldsymbol{e}_3'\boldsymbol{e}_4' & : \quad i = 1 \\ +g_2\boldsymbol{e}_1'\boldsymbol{e}_3'\boldsymbol{e}_4' & : \quad i = 2 \\ -g_3\boldsymbol{e}_1'\boldsymbol{e}_2'\boldsymbol{e}_4' & : \quad i = 3 \\ +g_4\boldsymbol{e}_1'\boldsymbol{e}_2'\boldsymbol{e}_3' & : \quad i = 4 \end{cases}
\end{aligned}$$

we write

$$\begin{aligned}
\boldsymbol{G} = S'\,\mathbf{I} &+ V'^1\boldsymbol{e}_1' + V'^2\boldsymbol{e}_2' + V'^3\boldsymbol{e}_3' + V'^4\boldsymbol{e}_4'\\
&+ T'^{12}\boldsymbol{e}_{12}' + T'^{13}\boldsymbol{e}_{13}' + T'^{14}\boldsymbol{e}_{14}' + T'^{23}\boldsymbol{e}_{23}' + T'^{24}\boldsymbol{e}_{24}' + T'^{34}\boldsymbol{e}_{34}'\\
&+ A'^1\boldsymbol{f}_1' + A'^2\boldsymbol{f}_2' + A'^3\boldsymbol{f}_3' + A'^4\boldsymbol{f}_4'\\
&+ P'\boldsymbol{f}'\\
\boldsymbol{H} = \text{same} &\text{ with lower case coefficients}
\end{aligned}$$

and work out *by hand* all the $16^2 = 256$ terms that enter into the construction of $\boldsymbol{G}\cdot\boldsymbol{H}$.[47] We are led to a result that can be written

[47] This is fairly easy: there are only $5^2 = 25$ categories of terms, and 10 of those are trivial (contain $\mathbf{I}$ as a factor). Patterns present within each category help one spot careless errors. Some typical calculations:

$$\begin{aligned}
T'^{12}v'^1\boldsymbol{e}_{12}'\boldsymbol{e}_1' &= -T'^{12}v'^1\boldsymbol{e}_1'\boldsymbol{e}_1'\boldsymbol{e}_2' = -g_1T'^{12}v'^1\boldsymbol{e}_2' = -v_1'T'^{12}\boldsymbol{e}_2'\\
T'^{12}v'^3\boldsymbol{e}_{12}'\boldsymbol{e}_3' &= \quad T'^{12}v'^3\boldsymbol{e}_1'\boldsymbol{e}_2'\boldsymbol{e}_3' = \quad (1/g_4)T'^{12}v'^3\boldsymbol{f}_4'\\
&= T'^{12}v'^3\boldsymbol{f}'^4\\
A'^1v'^1\boldsymbol{f}_1'\boldsymbol{e}_1' &= -g_1A'^1v'^1\boldsymbol{e}_2'\boldsymbol{e}_3'\boldsymbol{e}_4'\boldsymbol{e}_1' = +g_1A'^1v'^1\boldsymbol{f}'\\
A'^1v'^2\boldsymbol{f}_1'\boldsymbol{e}_2' &= -g_1A'^1v'^2\boldsymbol{e}_2'\boldsymbol{e}_3'\boldsymbol{e}_4'\boldsymbol{e}_2' = -g_1g_2A'^1v'^2\boldsymbol{e}_{34}' = -(g/g_3g_4)A'^1v'^2\boldsymbol{e}_{34}'\\
&= -gA'^1v'^2\boldsymbol{e}'^{34}
\end{aligned}$$

Here $g = g_1g_2g_3g_4$ is the determinant of the diagonal metric, and g_i^{-1} are the diagonal elements of its inverse. Dangling g_i and g_i^{-1} factors have been absorbed into index lowering/raising procedures of the sort standard to tensor algebra.

$$\begin{aligned}
\boldsymbol{G}\cdot\boldsymbol{H} = {} & \mathbf{I}\Big\{Ss + V_m v^m - \tfrac{1}{2}T_{mn}t^{mn} - gA_n a^n + gPp\Big\}\\
& + \boldsymbol{e}_i\Big\{(Sv^i + sV^i) + (V_m t^{mi} - v_m T^{mi})\\
& \qquad\qquad - \tfrac{1}{2}\varepsilon^{ilmn}(A_l t_{mn} + a_l T_{mn}) + g(Pa^i - pA^i)\Big\}\\
& + \tfrac{1}{2}\boldsymbol{e}_{ij}\Big\{(St^{ij} + sT^{ij}) + (V^i v^j - V^j v^i)\\
& \qquad\qquad + \tfrac{1}{2}\varepsilon^{ijmn}(V_m a_n - V_n a_m)\\
& \qquad\qquad - (T^i{}_m t^{jm} - T^j{}_m t^{im})\\
& \qquad\qquad - \tfrac{1}{2}\varepsilon^{ijmn}(A_m v_n - A_n v_m)\\
& \qquad\qquad - g(A^i a^j - A^j a^i)\\
& \qquad\qquad - \tfrac{1}{2}\varepsilon^{ijmn}(Pt_{mn} + pT_{mn})\Big\}\\
& + \boldsymbol{f}_j\Big\{(Sa^j + sA^j) - \tfrac{1}{2g}\varepsilon^{jlmn}(V_l t_{mn} + v_l T_{mn})\\
& \qquad\qquad + (A_m t^{mj} - a_m T^{mj}) + (Pv^j - pV^j)\Big\}\\
& + \boldsymbol{f}\Big\{Sp - V_m a^m + \tfrac{1}{4g}\varepsilon^{klmn}T_{kl}t_{mn} + A_n v^n + Ps\Big\}
\end{aligned} \tag{153}$$

where <u>all coefficients and Clifford elements are understood to wear primes</u>. Index raising/lowering manipulations have used to put all g_i-factors discretely out of sight: they survive only in the combination $g \equiv g_1 g_2 g_3 g_4$.

Familiarly, every real symmetric $\mathbb{g}$ can be rotated to diagonal form, and the numbers that then stand on the diagonal are the eigenvalues of $\mathbb{g}$. We will use that technique—but in reverse—to relax the diagonality assumption that was basic to the derivation of (153). To that end, introduce new/alternative generators $\boldsymbol{e}_i$ that are real linear combinations of the old ones (and *vice versa*):

$$\boldsymbol{e}'_i = R^p{}_i \boldsymbol{e}_p \tag{154}$$

Then (152) becomes

$$R^p{}_i R^q{}_j(\boldsymbol{e}_p\boldsymbol{e}_q + \boldsymbol{e}_q\boldsymbol{e}_p) = 2g'_{ij}\,\mathbf{I}$$

Let $\mathbb{S} \equiv \|S^i{}_m\|$ denote the inverse of $\mathbb{R} \equiv \|R^p{}_i\|$: $R^p{}_i S^i{}_m = \delta^p{}_m$. Then

$$\boldsymbol{e}_m\boldsymbol{e}_n + \boldsymbol{e}_n\boldsymbol{e}_m = 2g_{mn}\,\mathbf{I} \quad \text{with} \quad g_{mn} \equiv S^i{}_m S^j{}_n g'_{ij}$$
$$\updownarrow$$
$$\mathbb{g} = \mathbb{S}^\top \mathbb{g}'\,\mathbb{S}$$

The matrices $\mathbb{g}$ and $\mathbb{g}'$ will be spectrally identical if and only if $\mathbb{S}^\top = \mathbb{S}^{-1}$, which is to say: if and only if $\mathbb{S}$—whence also $\mathbb{R}$, its inverse—is a rotation matrix,[48]

[48] This means that the vectors $\boldsymbol{R}_i$ assembled from the respective columns of $\mathbb{R}$—the components of which appear as coefficients in the i^{th} instance of (154)—are *orthonormal*. For the purposes at hand they can be *any* such set of vectors.

...which we will henceforth assume to be the case. From (154) it follows that

$$V'^i \boldsymbol{e}'_i = V'^i R^p{}_i \boldsymbol{e}_p = V^p \boldsymbol{e}_p$$

provided V'^i transforms (as we will assume) as a contravariant vector:

$$V^p = R^p{}_i V'^i$$

Removal of the primes from $\boldsymbol{e}'_1\boldsymbol{e}'_2$, $\boldsymbol{e}'_1\boldsymbol{e}'_3$, ...involves a subtlty traceable to the circumstance that when we resolve $\boldsymbol{e}'_i\boldsymbol{e}'_j$ into its symmetric/antisymmetric parts

$$\boldsymbol{e}'_i\boldsymbol{e}'_j = \tfrac{1}{2}(\boldsymbol{e}'_i\boldsymbol{e}'_j + \boldsymbol{e}'_j\boldsymbol{e}'_i) + \tfrac{1}{2}(\boldsymbol{e}'_i\boldsymbol{e}'_j - \boldsymbol{e}'_j\boldsymbol{e}'_i)$$

the symmetric part $\frac{1}{2}(\boldsymbol{e}'_i\boldsymbol{e}'_j + \boldsymbol{e}'_j\boldsymbol{e}'_i) = g'_{ij}\,\mathsf{I}$ *vanishes* $(i \neq j)$ owing to the assumed diagonality of g'_{ij}. Its transform $\frac{1}{2}(\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i) = g_{ij}\,\mathsf{I}$—already discussed—is fundamental, but is always proportional to the transformationally invariant object I. Reference to "the transform of $\boldsymbol{e}'_{ij} = \boldsymbol{e}'_i\boldsymbol{e}'_j$ $(i \neq j)$" must be understood as a reference to the antisymmetric part of $\boldsymbol{e}'_{ij}$:

$$\begin{aligned} \boldsymbol{e}'_{ij} \equiv \tfrac{1}{2}(\boldsymbol{e}'_i\boldsymbol{e}'_j - \boldsymbol{e}'_j\boldsymbol{e}'_i) &= R^p{}_i R^q{}_j \tfrac{1}{2}(\boldsymbol{e}_p\boldsymbol{e}_q - \boldsymbol{e}_q\boldsymbol{e}_p) \\ &\qquad \tfrac{1}{2}(\boldsymbol{e}_p\boldsymbol{e}_q - \boldsymbol{e}_q\boldsymbol{e}_p) = \boldsymbol{e}_p\boldsymbol{e}_q - g_{pq}\,\mathsf{I} \equiv \boldsymbol{e}_{pq} \end{aligned}$$

We then have

$$T'^{ij}\boldsymbol{e}'_{ij} = T^{pq}\boldsymbol{e}_{pq}$$

if T'^{ij} is understood to transform as an antisymmetric contravariant tensor of second rank:

$$T^{pq} = R^p{}_i R^q{}_j T'^{ij}$$

To remove the primes from the totally antisymmetric object $\boldsymbol{e}'_1\boldsymbol{e}'_2\boldsymbol{e}'_3\boldsymbol{e}'_4$ we write $\boldsymbol{f}' = \frac{1}{4!}\varepsilon^{ijkl}\boldsymbol{e}'_i\boldsymbol{e}'_j\boldsymbol{e}'_k\boldsymbol{e}'_l$ as a way of automating the requirement that all indices be distinct (thus excluding products of—say—the type $\boldsymbol{e}'_1\boldsymbol{e}'_1\boldsymbol{e}'_3\boldsymbol{e}'_4$) and by transformation to unprimed generators obtain

$$\begin{aligned} \boldsymbol{f}' &= \tfrac{1}{4!}\varepsilon^{ijkl} R^p{}_i R^q{}_j R^r{}_k R^s{}_l\, \boldsymbol{e}_p\boldsymbol{e}_q\boldsymbol{e}_r\boldsymbol{e}_s \\ &= R\cdot\tfrac{1}{4!}\varepsilon^{pqrs}\boldsymbol{e}_p\boldsymbol{e}_q\boldsymbol{e}_r\boldsymbol{e}_s \quad \text{with} \quad R \equiv \det\mathbb{R} = \pm 1 \\ &= R\cdot\boldsymbol{f} \end{aligned}$$

We then have

$$P'\boldsymbol{f}' = P\boldsymbol{f}$$

provided P' is understood to transform as a scalar density of weight $W = -1$:

$$P = R\cdot P' \quad \text{which is to say:} \quad P' = R^{-1}\cdot P$$

Looking finally to $\boldsymbol{f}'_j \equiv \boldsymbol{f}'\boldsymbol{e}'_j$, it follows immediately from results now in hand that

$$\boldsymbol{f}'_j = R\cdot R^q{}_j\,\boldsymbol{f}_q \quad\text{with}\quad \boldsymbol{f}_q \equiv \boldsymbol{f}\boldsymbol{e}_q$$

from which we obtain

$$A'^j\boldsymbol{f}'_j = A^q\boldsymbol{f}_q$$

provided A'^j is understood to transform as a contravariant vector density of weight $W = -1$:

$$A^q = R\cdot R^q{}_j A'^j \quad\text{which is to say:}\quad A'^j = R^{-1}\cdot S^j{}_q A^q$$

The preceding argument establishes the structural invariance of (153) under all transformations that preserve the prescribed eigenvalues of $\mathfrak{g}$. But (153) holds *whatever those eigenvalues might be*. We conclude that the product formula is of <u>unrestricted generality</u>. Noting that

- g transforms as an object of weight $W = +2$
- ε^{ijkl}, $\boldsymbol{f}$ and $\boldsymbol{f}_j$ transform as objects of weight $W = +1$
- S, s, V^i, v^i, T^{ij}, t^{ij}, $\boldsymbol{e}_i$ and $\boldsymbol{e}_{ij}$ transform as objects of weight $W = 0$
- A^j, a^j, P and p transform as objects of weight $W = -1$
- g^{-1} transforms as an object of weight $W = -2$

we observe that

- the coefficient of $\mathbf{I}$ in (153) presents **all weightless scalars**—bilinear in the coefficients of $\boldsymbol{G}$ and $\boldsymbol{H}$—that can be assembled from the above material
- the coefficient of $\boldsymbol{e}_i$ in (153) presents **all weightless contravariant vectors** that can be assembled...
- the coefficient of $\boldsymbol{e}_{ij}$ in (153) presents **all weightless antisymmetric contravariant second rank tensors** that can be assembled...
- the coefficient of $\boldsymbol{f}_j$ in (153) presents **all contravariant vectors of negative unit weight** that can be assembled...
- the coefficient of $\boldsymbol{f}$ in (153) presents **all scalars of negative unit weight** that can be assembled...

Had we possessed this information in advance it would not, however, have permitted us to simply *write down* (153), for it speaks not at all to signs and numerics.

I turn now to discussion of some of the implications of (153).

7. Implications of the product formula. Given

$$\boldsymbol{G} = S\,\mathbf{I} + V^i\boldsymbol{e}_i + \tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij} + A^j\boldsymbol{f}_j + P\boldsymbol{f}$$

we introduce "conjugates" of two flavors:

$$\boldsymbol{G}^{\mathsf{T}} = S\,\mathbf{I} - V^i\boldsymbol{e}_i - \tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij} + A^j\boldsymbol{f}_j + P\boldsymbol{f} \tag{155.1}$$

$$\boldsymbol{G}^{\mathsf{t}} = S\,\mathbf{I} + V^i\boldsymbol{e}_i + \tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij} - A^j\boldsymbol{f}_j - P\boldsymbol{f} \tag{155.2}$$

It then follows as a corollary of (153) that

$$\begin{aligned}\boldsymbol{G}\boldsymbol{G}^{\mathsf{T}} = \;& \mathbf{I}\Big\{SS - V_mV^m + \tfrac{1}{2}T_{mn}T^{mn} - gA_nA^n + gPP\Big\}\\ &+\boldsymbol{f}_j\Big\{2SA^j + \tfrac{1}{g}\varepsilon^{jlmn}V_lT_{mn} - 2A_mT^{mj} - 2PV^j\Big\}\\ &+\boldsymbol{f}\,\Big\{2SP - 2V_mA^m - \tfrac{1}{4g}\varepsilon^{klmn}T_{kl}T_{mn}\Big\}\end{aligned} \tag{156}$$

from which—remarkably—all $\boldsymbol{e}_i$ and $\boldsymbol{e}_{ij}$ terms have vanished: we are left with an expression of what I will call the "pseudo-simple" form[49]

$$\mathcal{G} \equiv \mathcal{S}\,\mathbf{I} + \mathcal{A}^j\boldsymbol{f}_j + \mathcal{P}\boldsymbol{f} \tag{157}$$

which entails—again as a quick corollary of (153)—that $\mathcal{G}\mathcal{G}^{\mathsf{t}} = \mathcal{G}^{\mathsf{t}}\mathcal{G}$ is a simple multiple of $\mathbf{I}$. Specifically

$$\mathcal{G}\mathcal{G}^{\mathsf{t}} = \mathcal{N}(\boldsymbol{G})\,\mathbf{I} \tag{158}$$

where[50]

$$\mathcal{N}(\boldsymbol{G}) \equiv \mathcal{S}^2 + g\mathcal{A}^j\mathcal{A}_j - g\mathcal{P}^2 \tag{159}$$

defines what I will call the "norm" of $\boldsymbol{G}$. Evidently

$$\boldsymbol{G}^{-1} = \frac{1}{\mathcal{N}(\boldsymbol{G})}\,\boldsymbol{G}^{\mathsf{T}}(\boldsymbol{G}\boldsymbol{G}^{\mathsf{T}})^{\mathsf{t}} \;\;:\;\; \text{exists if and only if } \mathcal{N}(\boldsymbol{G}) \neq 0 \tag{160}$$

It was in prospect of this important result that the operations $^{\mathsf{T}}$ and $^{\mathsf{t}}$ were introduced, and I digress now to review their basic properties:

It is immediately evident that $^{\mathsf{T}}$ and $^{\mathsf{t}}$ are both *linear* operations, that they *commute*

$$(\boldsymbol{G}^{\mathsf{T}})^{\mathsf{t}} = (\boldsymbol{G}^{\mathsf{t}})^{\mathsf{T}} \tag{161.1}$$

and that

$$(\boldsymbol{G}^{\mathsf{T}})^{\mathsf{T}} = \boldsymbol{G} \quad : \quad (\boldsymbol{G}^{\mathsf{t}})^{\mathsf{t}} = \boldsymbol{G} \tag{161.2}$$

It is a (not so immediately evident) implication of (153) that

$$(\boldsymbol{G}\boldsymbol{H})^{\mathsf{T}} = \boldsymbol{H}^{\mathsf{T}}\boldsymbol{G}^{\mathsf{T}} \tag{161.3}$$

In this respect $^{\mathsf{T}}$ mimics a familiar property of the *transposition* operation, and we are brought by this remark to the realization that "pseudo-simplicity" and

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[49] It is perhaps worth noting that if $\mathcal{G}$ and $\mathcal{H}$ are pseudo-simple their product is, in general, *not* pseudo-simple.

[50] Compare (77) on page 20.

and "symmetry"—in the sense "invariant under the action of ${}^{\intercal}$"—are equivalent notions: the Clifford numbers $\frac{1}{2}(\boldsymbol{G}+\boldsymbol{G}^{\intercal})$, $\boldsymbol{G}\boldsymbol{G}^{\intercal}$ and $\boldsymbol{G}^{\intercal}\boldsymbol{G}$ are all pseudo-simple, all sent into themselves by ${}^{\intercal}$. It becomes interesting in this light to observe that it is generally the case—even when $\boldsymbol{G}$ and $\boldsymbol{H}$ are both pseudo-simple—that (according to (153))

$$(\boldsymbol{G}\boldsymbol{H})^{\mathsf{t}} = \text{neither } \boldsymbol{H}^{\mathsf{t}}\boldsymbol{G}^{\mathsf{t}} \text{ nor } \boldsymbol{G}^{\mathsf{t}}\boldsymbol{H}^{\mathsf{t}}$$

I describe now some algebraic problems that, while they lie near the heart of the theory, we seem to be not yet in position to attack. Equation (160)—which can be formulated

$$\boldsymbol{G}\boldsymbol{G}^{\intercal}(\boldsymbol{G}\boldsymbol{G}^{\intercal})^{\mathsf{t}} = \mathcal{N}(\boldsymbol{G})\,\mathbf{I}$$

—describes the *right* inverse of $\boldsymbol{G}$. Application of ${}^{\intercal}$ gives

$$(\boldsymbol{G}\boldsymbol{G}^{\intercal})^{\mathsf{t}}\,\boldsymbol{G}\boldsymbol{G}^{\intercal} = \mathcal{N}(\boldsymbol{G})\,\mathbf{I}$$

(according to which $\boldsymbol{G}\boldsymbol{G}^{\intercal}$ and $(\boldsymbol{G}\boldsymbol{G}^{\intercal})^{\mathsf{t}}$ *commute*) which, since valid for all $\boldsymbol{G}$, must remain valid when $\boldsymbol{G}$ is replaced by $\boldsymbol{G}^{\intercal}$:

$$(\boldsymbol{G}^{\intercal}\boldsymbol{G})^{\mathsf{t}}\,\boldsymbol{G}^{\intercal}\boldsymbol{G} = \mathcal{N}(\boldsymbol{G}^{\intercal})\,\mathbf{I}$$

Evidently

$$\left.\begin{aligned}\text{right inverse of } \boldsymbol{G} &= \frac{1}{\mathcal{N}(\boldsymbol{G})}\,\boldsymbol{G}^{\intercal}(\boldsymbol{G}\boldsymbol{G}^{\intercal})^{\mathsf{t}}\\ \text{left inverse of } \boldsymbol{G} &= \frac{1}{\mathcal{N}(\boldsymbol{G}^{\intercal})}(\boldsymbol{G}^{\intercal}\boldsymbol{G})^{\mathsf{t}}\boldsymbol{G}^{\intercal}\end{aligned}\right\}\tag{162}$$

We know on general grounds[51] that the expressions on the right side of (162) must be equal, but are not presently in position to argue that they are "obviously" so. If by brute force appeal to (153) we could show that

$$\boldsymbol{G}^{\intercal}(\boldsymbol{G}\boldsymbol{G}^{\intercal})^{\mathsf{t}} = (\boldsymbol{G}^{\intercal}\boldsymbol{G})^{\mathsf{t}}\boldsymbol{G}^{\intercal}$$

(which in the Euclidean case I have, with the assistance of *Mathematica*, actually done) one would have

$$\mathcal{N}(\boldsymbol{G}) = \mathcal{N}(\boldsymbol{G}^{\intercal})\tag{163}$$

which would appear to be even harder (sixteen times harder) to establish by brute force calculation. Arguing similarly from $(\boldsymbol{G}\boldsymbol{H})^{-1} = \boldsymbol{H}^{-1}\boldsymbol{G}^{-1}$ we expect to have

$$\frac{1}{\mathcal{N}(\boldsymbol{G}\boldsymbol{H})}\,\boldsymbol{H}^{\intercal}\boldsymbol{G}^{\intercal}(\boldsymbol{G}\boldsymbol{H}\boldsymbol{H}^{\intercal}\boldsymbol{G}^{\intercal})^{\mathsf{t}} = \frac{1}{\mathcal{N}(\boldsymbol{G})\mathcal{N}(\boldsymbol{H})}\,\boldsymbol{H}^{\intercal}(\boldsymbol{H}\boldsymbol{H}^{\intercal})^{\mathsf{t}}\,\boldsymbol{G}^{\intercal}(\boldsymbol{G}\boldsymbol{G}^{\intercal})^{\mathsf{t}}$$

[51] If $AX = XB = 1$ then multiplication by B on the right supplies $A = B$.

which—if we could (whether by frontal attack or by indirection) establish

$$\boldsymbol{H}^{\mathsf{T}}\boldsymbol{G}^{\mathsf{T}}(\boldsymbol{G}\boldsymbol{H}\boldsymbol{H}^{\mathsf{T}}\boldsymbol{G}^{\mathsf{T}})^{\mathsf{t}} = \boldsymbol{H}^{\mathsf{T}}(\boldsymbol{H}\boldsymbol{H}^{\mathsf{T}})^{\mathsf{t}}\,\boldsymbol{G}^{\mathsf{T}}(\boldsymbol{G}\boldsymbol{G}^{\mathsf{T}})^{\mathsf{t}}$$

—would entail that $\mathcal{N}(\boldsymbol{G})$ possesses the "determinantal property"

$$\mathcal{N}(\boldsymbol{G}\boldsymbol{H}) = \mathcal{N}(\boldsymbol{G})\,\mathcal{N}(\boldsymbol{H}) \tag{164}$$

It is clear from its definition (159) that $\mathcal{N}(\boldsymbol{G})$ is a multinomial of 4^{th} order in the coefficients of $\boldsymbol{G}$. I undertake here to develop its explicit structure. Looking to (156/157) we see that $\mathcal{S}$, $\mathcal{A}^j$ and $\mathcal{P}$ can be described

$$\left.\begin{aligned} \mathcal{S} &= \quad S^2 + \sigma \\ \mathcal{A}^j &= \alpha^j S \;\;+ \beta^j \\ \mathcal{P} &= \pi\;\; S \;\;+ \rho \end{aligned}\right\} \tag{165}$$

where I have isolated the S-dependence for reasons that will soon emerge. In this notation

$$\begin{aligned} \mathcal{N}(\boldsymbol{G}) &= (S^2+\sigma)^2 + g(\alpha^j S + \beta^j)(\alpha_j S + \beta_j) - g(\pi S + \rho)^2 \\ &= S^4 + (2\sigma + g\alpha^j\alpha_j - g\pi^2)S^2 + 2g(\alpha^j\beta_j - \pi\rho)S + (\sigma^2 + g\beta^j\beta_j - g\rho^2) \\ &\equiv S^4 + (\text{no } S^3\text{-term}) + N_2 S^2 + N_3 S^1 + N_4 S^0 \end{aligned} \tag{166}$$

This result puts us in position to develop $\mathcal{N}(\boldsymbol{G}-\lambda\mathbf{I})$ in powers of λ:

$$\begin{aligned} \mathcal{N}(\boldsymbol{G}-\lambda\mathbf{I}) = \lambda^4 - 4S\lambda^3 &+ (6S^2 + N_2)\lambda^2 \\ &- (4S^3 + 2N_2 S + N_3)\lambda \\ &+ (S^4 + N_2 S^2 + N_3 S + N_4) \\ \equiv \lambda^4 + \mathcal{N}_3\lambda^3 &+ \mathcal{N}_2\lambda^2 + \mathcal{N}_1\lambda^1 + \mathcal{N}_0\lambda^0 \end{aligned} \tag{167}$$

Tentatively $\boxed{\text{assuming } \mathcal{N}(\boldsymbol{G}\boldsymbol{H}) = \mathcal{N}(\boldsymbol{G})\,\mathcal{N}(\boldsymbol{H}) \text{ to have been established}}$, we have the similarity-transform invariance of the norm

$$\mathcal{N}(\boldsymbol{U}^{-1}\boldsymbol{G}\boldsymbol{U}) = \mathcal{N}(\boldsymbol{G}) \tag{168}$$

which implies the similarity-transform invariance of $\{\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$. And this—by *Mathematica*-assisted inversion of the equations that describe the $\mathcal{N}$'s in terms of the N's, a process that supplies

$$\begin{aligned} S &= -\tfrac{1}{4}\mathcal{N}_3 \\ N_2 &= \quad\;\mathcal{N}_2 - \tfrac{3}{8}\mathcal{N}_3^2 \\ N_3 &= -\;\;\mathcal{N}_1 + \tfrac{1}{2}\mathcal{N}_2\mathcal{N}_3 - \tfrac{1}{8}\mathcal{N}_3^3 \\ N_4 &= \quad\;\mathcal{N}_0 - \tfrac{1}{4}\mathcal{N}_1\mathcal{N}_3 + \tfrac{1}{16}\mathcal{N}_2\mathcal{N}_3^2 - \tfrac{3}{256}\mathcal{N}_3^4 \end{aligned}$$

—implies (as it is also implied by) the invariance of $\{S, N_0, N_1, N_2\}$. And by formal extension of the Cayley-Hamilton theorem we expect $\boldsymbol{G}$ itself to be a solution of its own characteristic equation:

$$\boldsymbol{G}^4 + \mathcal{N}_3\boldsymbol{G}^3 + \mathcal{N}_2\boldsymbol{G}^2 + \mathcal{N}_1\boldsymbol{G}^1 + \mathcal{N}_0\boldsymbol{G}^0 = \mathbf{0} \tag{169}$$

Direct computational varification of this statement—imitative of what within $\mathcal{C}_2[\boldsymbol{g}]$ was accomplished at (79) on page 20—would appear, however, to lie far beyond the bounds of feasibility.

At (165) we found it convenient to introduce these abbreviations:

$$\begin{aligned}
\sigma &\equiv -V_m V^m + \tfrac{1}{2}T_{mn}T^{mn} - gA_nA^n + gPP\\
\alpha^j &\equiv 2A^j\\
\beta^j &\equiv \tfrac{1}{g}\varepsilon^{jlmn}V_l T_{mn} - 2A_mT^{mj} - 2PV^j\\
\pi &\equiv 2P\\
\rho &\equiv -2V_mA^m - \tfrac{1}{4g}\varepsilon^{klmn}T_{kl}T_{mn}
\end{aligned}$$

Returning with that information to (166) we obtain

$$\left.\begin{aligned}
N_2 &= -2V_mV^m - T_{mn}T^{nm} + 2gA_nA^n - 2gP^2\\
\\
N_3 &= P\varepsilon^{klmn}\big[T_{kl} + 2(A_kV_l - A_lV_k)\big]T_{mn}\\
\\
N_4 &= \tfrac{1}{4}(T_{mn}T^{nm})^2 - \tfrac{1}{16g}(\varepsilon^{klmn}T_{kl}T_{mn})^2\\
&\qquad\qquad + (T_{mn}T^{nm})(V_kV^k + gA_kA^k)\\
&\qquad\qquad + \tfrac{1}{g}\varepsilon^{klmn}V_lT_{mn}\varepsilon_{krst}V^rT^{st}\\
&\qquad\qquad - 4gA_mT^{mk}T_{kn}A^n\\
&\qquad\qquad + (V_kV^k)^2 + g^2(A_kA^k)^2\\
&\qquad\qquad - 4g(V_kA^k)^2 + 2g(V_jV^j)(A_kA^k)\\
&\qquad\qquad + 8gP(A_mT^{mn}V_n) - gP^2(T_{mn}T^{nm})\\
&\qquad\qquad + 2gP^2(V_kV^k - gA_kA^k) + g^2P^4
\end{aligned}\right\} \tag{170}$$

after simplifications.[52] My notation has been designed to underscore a fact now evident:

$$N_k \text{ is homogeneous of degree } k \text{ in } \big\{V, T, A, P\big\}$$

The terms that enter into the description of N_4 can be grouped in a great variety of "natural" ways: which is most useful was found in the GENEVA NOTEBOOK to depend upon the context. It became obvious at (166) that

$$\mathcal{N}(\boldsymbol{G}) = N_4 \quad \text{in cases where} \quad S = 0$$

[52] See the GENEVA NOTEBOOK, page 209.

Notice that the expressions on the right side of (170) are invariant under $\{S, V, T, A, P\} \rightarrow \{S, -V, -T, A, P\}$. We can on this basis consider (163) to be an established fact:[53] $\mathcal{N}(\boldsymbol{G}) = \mathcal{N}(\boldsymbol{G}^{\mathsf{T}})$.

We are, however, no closer than before to proof of the conjectured identity (164), upon which our recent remarks are critically dependent. Proof would be immediate if we could set up a **matrix representation** of $\mathcal{C}_4[\boldsymbol{g}]$

$$\boldsymbol{G} \longleftrightarrow \mathbb{G}$$

within which[54]

$$\mathcal{N}(\boldsymbol{G}) = \det \mathbb{G}$$

A procedure that might in principle work proceeds from the fact that $\mathcal{C}_4[\boldsymbol{g}]$ is *associative*: if we were (i) to

$$\text{write } \boldsymbol{G} = \sum_{p=0}^{15} G^p \boldsymbol{e}_p \text{ in place of } \boldsymbol{G} = S\,\mathbf{I} + V^i \boldsymbol{e}_i + \tfrac{1}{2}T^{ij}\boldsymbol{e}_{ij} + A^j \boldsymbol{f}_j + P\boldsymbol{f}$$

then (ii) to work out the values of the $16^3 = 4096$ (real-valued) structure constants ${c_p}^r{}_q$ that enter into the statements

$$\boldsymbol{e}_p \boldsymbol{e}_q = \sum_{r=0}^{15} {c_p}^r{}_q \boldsymbol{e}_r$$

and (iii) used them to assemble 16×16 real matrices $\mathbb{E}_p \equiv \|{c_p}^r{}_q\|$ we would—as an expression of $(\boldsymbol{e}_p\boldsymbol{e}_q)\boldsymbol{e}_s = \boldsymbol{e}_p(\boldsymbol{e}_q\boldsymbol{e}_s)$—arrive at the "regular representation"

$$\mathbb{E}_p \mathbb{E}_q = \sum_{r=0}^{15} {c_p}^r{}_q \mathbb{E}_r$$

The demonstration that if

$$\mathbb{G} = \sum_{p=0}^{15} G^p\,\mathbb{E}_p \quad \text{represents} \quad \boldsymbol{G} = \sum_{p=0}^{15} G^p \boldsymbol{e}_p$$

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[53] It is, on the other hand, *not* generally the case that $\mathcal{N}(\boldsymbol{G}) = \mathcal{N}(\boldsymbol{G}^{\mathbf{t}})$: N_0 and N_2 are invariant under $\{S, V, T, A, P\} \rightarrow \{S, V, T, -A, -P\}$ but N_1 reverses its sign. If, however, $\boldsymbol{G}$ is pseudo-simple (*i.e.*, if $V = T = 0$) then $N_1 = 0$. The short of it: we have $\mathcal{N}(\boldsymbol{G}) = \mathcal{N}(\boldsymbol{G}^{\mathbf{t}})$ if and only if $\boldsymbol{G}$ is pseudo-simple.

[54] A relation of the weaker form

$$[\mathcal{N}(\boldsymbol{G})]^{\text{characteristic power}} = \det \mathbb{G}$$

would serve just as well.

then

$$\det \mathbb{G} = \begin{cases} [\mathcal{N}(\boldsymbol{G})]^1, \text{ else} \\ [\mathcal{N}(\boldsymbol{G})]^2, \text{ else} \\ [\mathcal{N}(\boldsymbol{G})]^4 \end{cases}$$

would appear, however, to be enormously tedious (and for that very reason not deeply instructive).

Because $\mathcal{N}(\boldsymbol{G})$ is of order 4 in the coeffients of $\boldsymbol{G}$ it becomes natural to look for 4×4 matrices $\mathbb{E}_p$, for then $\det \mathbb{G}$ would also be of order 4. Under favorable circumstances that I will, for the moment, not attempt to characterize it may happen that

$$\det \mathbb{G} \text{ is invariably real, even though } \mathbb{G} \text{ is complex.}$$

It becomes then feasible that $\det \mathbb{G} = \mathcal{N}(\boldsymbol{G})$. I illustrate how this works out in the case $\mathcal{C}_4[\boldsymbol{g}_{\text{Euclidean}}]$. Let the generators $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be represented by "Euclideanized" variants of the Dirac matrices encountered on page 57:

$$\mathbb{E}_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mathbb{E}_2 \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{E}_3 \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{E}_4 \equiv \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

Those matrices happen to be traceless hermitian, and (as it turns out) so is the implied representative of $\boldsymbol{f}$. The representatives of $\boldsymbol{e}_{ij}$ and $\boldsymbol{f}_j$ are found, however, to be traceless *anti*hermitian: the representative $\mathbb{G}$ of $\boldsymbol{G}$ is therefore curiously non-descript! Nevertheless... *Mathematica* responds to the command

```
Det[𝔾]//ComplexExpand
```

with many lines of manifestly real output. Turning our attention now to the representation $\mathbb{G}\mathbb{G}^{\intercal}(\mathbb{G}\mathbb{G}^{\intercal})^{\mathsf{t}}$—which we intend to compare with $\mathcal{N}(\boldsymbol{G})\mathsf{I}$—given

$$\mathbb{G} \;= S\,\mathbb{I} + V^i\,\mathbb{E}_i + \tfrac{1}{2}T^{ij}\,\mathbb{E}_{ij} + A^j\,\mathbb{F}_j + P\,\mathbb{F}$$

it is easy enough to construct

$$\mathbb{G}^{\intercal} = S\,\mathbb{I} - V^i\,\mathbb{E}_i - \tfrac{1}{2}T^{ij}\,\mathbb{E}_{ij} + A^j\,\mathbb{F}_j + P\,\mathbb{F}$$

and to compute $\mathbb{G}.\mathbb{G}^{\intercal}$, but the output is a formless mess. How to demonstrate that it has the form $\mathcal{S}\,\mathbb{I} + \mathcal{A}^j\,\mathbb{F}_j + \mathcal{P}\,\mathbb{F}$? And how to construct $\mathcal{S}\,\mathbb{I} - \mathcal{A}^j\,\mathbb{F}_j - \mathcal{P}\,\mathbb{F}$? The first question is resolved by appeal to the fact that

$$\tfrac{1}{4}\text{tr}\,(\mathbb{E}_p\mathbb{E}_q) = \pm\delta_{pq} \quad : \quad \begin{cases} \text{upper sign for } \mathbb{I},\ \mathbb{E}_i \text{ and } \mathbb{F} \\ \text{lower sign for } \mathbb{E}_{ij} \text{ and } \mathbb{F}_j \end{cases} \tag{171}$$

For it is a computationally demonstrable property of the "formless mess" that

$$\tfrac{1}{4}\text{tr}\,(\mathbb{E}_i\mathbb{G}\,.\,\mathbb{G}^\intercal) = \tfrac{1}{4}\text{tr}\,(\mathbb{E}_{ij}\mathbb{G}\,.\,\mathbb{G}^\intercal) = 0$$

so $\mathbb{G}\,.\,\mathbb{G}^\intercal$ does indeed have the form $\mathcal{S}\,\mathbb{I} + \mathcal{A}^j\,\mathbb{F}_j + \mathcal{P}\mathbb{F}$. Noting that

$$\mathcal{S}\,\mathbb{I} - \mathcal{A}^j\,\mathbb{F}_j - \mathcal{P}\mathbb{F} = 2\mathcal{S}\,\mathbb{I} + (\mathcal{S}\,\mathbb{I} + \mathcal{A}^j\,\mathbb{F}_j - \mathcal{P}\mathbb{F})$$

we construct

$$(\mathbb{G}\,.\,\mathbb{G}^\intercal)^{\mathsf{t}} = \tfrac{1}{2}\text{tr}\,(\mathbb{G}\,.\,\mathbb{G}^\intercal)\,\mathbb{I} - \mathbb{G}\,.\,\mathbb{G}^\intercal$$

(which is manageable on the computer, even though all the terms involved are gigantic). This done, we are informed that indeed

$$\tfrac{1}{4}\text{tr}\left\{(\mathbb{G}\,.\,\mathbb{G}^\intercal)\,.\,(\mathbb{G}\,.\,\mathbb{G}^\intercal)^{\mathsf{t}}\right\} = \det\mathbb{G} \tag{172}$$

So

$$\boldsymbol{G}\boldsymbol{G}^\intercal(\boldsymbol{G}\boldsymbol{G}^\intercal)^{\mathsf{t}} = \mathcal{N}(\boldsymbol{G})\,\mathbf{I} \tag{173.1}$$

has acquired the representation

$$\mathbb{G}\,.\,\mathbb{G}^\intercal.\,(\mathbb{G}\,.\,\mathbb{G}^\intercal)^{\mathsf{t}} = (\det\mathbb{G})\,\mathbb{I} \tag{173.2}$$

which establishes the point at issue. By the "rescale and rotate" procedure

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} g_1 & 0 & 0 & 0\\ 0 & g_2 & 0 & 0\\ 0 & 0 & g_3 & 0\\ 0 & 0 & 0 & g_4 \end{pmatrix} \longmapsto \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

developed on pages 64–68 we expect to be able to abandon the Euclidean metric presumption that entered this discussion with the construction of our $\mathbb{E}_i$ matrices, but I am presently disinclined to pursue those details.

8. Similarity transformations in the fourth order theory. We look now[55] to the norm-preserving transformations

$$\boldsymbol{G} \longmapsto \boldsymbol{G}' = \boldsymbol{U}^{-1}\boldsymbol{G}\boldsymbol{U} \tag{174}$$

Proceding initially on the assumption that $\boldsymbol{U}$ differs only infinitesimally from the identity

$$\boldsymbol{U} = \boldsymbol{I} + \epsilon\boldsymbol{H}$$

we have

$$\boldsymbol{G}' = \boldsymbol{G} + \epsilon[\boldsymbol{G}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{G}] + \cdots \tag{175}$$

[55] See again pages 22 *et seq* and page 56.

The product rule (153) supplies

$$\begin{aligned}
\boldsymbol{GH}-\boldsymbol{HG} = {}& 2\boldsymbol{e}_i\Big\{(V_m t^{mi}-v_m T^{mi})+g(Pa^i-pA^i)\Big\}\\
&+\boldsymbol{e}_{ij}\Big\{(V^i v^j - V^j v^i)-(T^i{}_m t^{jm}-T^j{}_m t^{im})-g(A^i a^j-A^j a^i)\Big\}\\
&+2\boldsymbol{f}_j\Big\{(A_m t^{mj}-a_m T^{mj})+(Pv^j-pV^j)\Big\}\\
&+2\boldsymbol{f}\Big\{A_m v^m - V_m a^m\Big\}\\
={}& 2\Big\{-t^i{}_m V^m - v_m T^{mi}-gpA^i+ga^iP\Big\}\boldsymbol{e}_i\\
&+\Big\{-(v^iV^j-v^jV^i)-(t^i{}_m T^{mj}-t^j{}_m T^{mi})+g(a^iA^j-a^jA^i)\Big\}\boldsymbol{e}_{ij}\\
&+2\Big\{-pV^j-a_mT^{mj}-t^j{}_m A^m+v^jP\Big\}\boldsymbol{f}_j\\
&+2\Big\{-a_mV^m+v_mA^m\Big\}\boldsymbol{f}
\end{aligned}\tag{176}$$

S, the coefficient of $\mathbf{I}$ in the development of $\boldsymbol{G}$, is similarity invariant: $S'=S$. We concentrate therefore on the relation of the 15 numbers $\{V',T',A',P'\}$ to their unprimed counterparts, and for this purpose me may as well—will—assume $\boldsymbol{G}$ to be "pure": $S=0$. Nor (as we have just seen) does its scalar part $s\mathbf{I}$ contribute to the action of $\boldsymbol{H}$, so we assume also that $\boldsymbol{H}$ is pure: $s=0$. Let the $\boldsymbol{G}$-coefficients be strung out as a column vector:

$$\vec{G}\equiv\begin{pmatrix} V^1\\ V^2\\ V^3\\ V^4\\ T^{12}\\ T^{13}\\ T^{12}\\ T^{14}\\ T^{23}\\ T^{24}\\ T^{34}\\ A^1\\ A^2\\ A^3\\ A^4\\ P\end{pmatrix}\tag{177}$$

It follows now from (176) that in this notation the infinitesimal similarity transformation (175) can be described

$$\vec{G}\longmapsto\vec{G}'=(\mathbb{I}+2\epsilon\mathbb{H}+\cdots)\vec{G}$$

where $\mathbb{I}$ is the 15×15 unit matrix and $\mathbb{H}$ is a 15×15 matrix the detailed structure of which can be read off from (176). By iteration we find (compare page 24) that

$$\boldsymbol{G} \longmapsto \boldsymbol{G}' = e^{-\theta \boldsymbol{H}} \boldsymbol{G} e^{\theta \boldsymbol{H}} \tag{178.1}$$

can be described

$$\vec{G} \longmapsto \vec{G}' = e^{2\theta\,\mathbb{H}}\,\vec{G} \tag{178.2}$$

The quadratic similarity-invariant N_2 (see again (170)) acquires in this notation a fairly natural description. Writing

$$-\tfrac{1}{2}N_2 = g_{mn}V^mV^n - \tfrac{1}{2}g_{mp}g_{nq}T^{mn}T^{pq} - g\cdot g_{mn}A^mA^n + g\cdot P^2$$

we have

$$= \vec{G}^{\top}\,\mathbb{M}\,\vec{G} \tag{179}$$

where $\mathbb{M}$ is a 15×15 symmetric matrix assembled from elements of the 4×4 metric matrix $\|g_{mn}\|$. In an obvious sense, $\mathbb{M}$ injects "induced metric structure" into 15-space. From the similarity invariance of $\vec{G}^{\top}\,\mathbb{M}\,\vec{G}$ we infer that $\mathbb{U}\equiv e^{2\theta\,\mathbb{H}}$ is "$\mathbb{M}$-orthogonal"

$$\mathbb{M}^{-1}\mathbb{U}^{\top}\mathbb{M} = \mathbb{U}^{-1}$$

and therefore that its logarithm $\mathbb{H}$ is "$\mathbb{M}$-antisymmetric"

$$\mathbb{M}^{-1}\mathbb{H}^{\top}\mathbb{M} = -\mathbb{H} \quad : \quad \mathbb{H}^{\top}\mathbb{M}\text{, therefore, is literally antisymmetric}$$

What to do about—what lesson is to be drawn from—the fact that the cubic and quartic invariants N_3 and N_4 find <u>no natural dwelling place within such a scheme</u>? These are the questions with which the work recorded in the GENEVA NOTEBOOK is largely concerned, and it is upon that work that I now draw. I will begin by describing the basic idea, then labor to develop the details in the instance that concerns us.

Let $\mathbb{M}\equiv\|m_{jk}\|$ be a non-singular symmetric $N\times N$ matrix, and let its inverse be denoted $\mathbb{W}\equiv\|w^{ij}\|$: $w^{ij}m_{jk}=\delta^i{}_k$. Let $\mathbb{A}\equiv\|a^j{}_k\|$ and, upon agreement that $\mathbb{W}$ and $\mathbb{M}$ will be used to raise/lower indices, write $\mathbb{M}\mathbb{A}\equiv\|a_{jk}\|$: $a_{jk}=m_{jp}a^p{}_k$. With the tensor rule $X_{ij}\longmapsto X'_{ij}=U^p{}_iU^q{}_jX_{pq}$ in mind we study transformations of the form

$$\mathbb{M}\mathbb{A}-\lambda\mathbb{M} \;\longmapsto\; (\mathbb{M}\mathbb{A}-\lambda\mathbb{M})' = \mathbb{U}^{\top}(\mathbb{M}\mathbb{A}-\lambda\mathbb{M})\mathbb{U} \tag{180}$$

with $\mathbb{U}\equiv\|U^j{}_k\|$. We now impose upon $\mathbb{U}$ the <u>restrictive assumption</u> that

$$\mathbb{M}' \equiv \mathbb{U}^{\top}\mathbb{M}\mathbb{U} = \mathbb{M} \quad : \quad \mathbb{U}\text{ is "}\mathbb{M}\text{-orthogonal"}$$

We then have $\mathbb{U}^{\top}(\mathbb{M}\mathbb{A}-\lambda\mathbb{M})\mathbb{U} = \mathbb{M}(\mathbb{U}^{-1}\mathbb{A}\mathbb{U}-\lambda\mathbb{I})$: the transformation (180), after multiplication on the left by $\mathbb{W}$, has assumed the form

$$\mathbb{A}-\lambda\mathbb{I} \;\longmapsto\; \mathbb{A}'-\lambda\mathbb{I} = \mathbb{U}^{-1}(\mathbb{A}-\lambda\mathbb{I})\mathbb{U} \tag{181}$$

of a *similarity transformation*. Immediately

$$\det(\mathbb{A}' - \lambda\mathbb{I}) = \det(\mathbb{A} - \lambda\mathbb{I})$$

The implication is that the coefficients Q_n that enter into the construction of the characteristic polynomial[56]

$$p(\lambda) \equiv \det(\mathbb{A} - \lambda\mathbb{I}) = \sum_{n=0}^{N} \tfrac{1}{n!}Q_n(-\lambda)^{N-n} \tag{182}$$

are $\mathbb{U}$-*invariant functions of the elements of* $\mathbb{A}$. I describe now two distinct methods for *constructing* the Q_n:

It is a fact—as little known as it is pretty—that Q_n can be described

$$Q_0 = 1$$

$$Q_n = \begin{vmatrix} T_1 & T_2 & T_3 & T_4 & \dots & T_n \\ 1 & T_1 & T_2 & T_3 & \dots & T_{n-1} \\ 0 & 2 & T_1 & T_2 & \dots & T_{n-2} \\ 0 & 0 & 3 & T_1 & \dots & T_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & T_1 \end{vmatrix} \quad : \quad n = 1, 2, 3, \dots \tag{183.1}$$

where

$$T_n \equiv \operatorname{tr}\mathbb{A}^n \tag{183.2}$$

We are led from this result to the recursion relation

$$Q_n = \sum_{m=1}^{n} (-)^{m+1}\frac{(n-1)!}{(n-m)!}T_m Q_{n-m}$$

from which it follows (with a little assistance from *Mathematica*) that

$$\begin{aligned}
Q_0 &= 1\\
Q_1 &= T_1\\
Q_2 &= T_1^2 - T_2\\
Q_3 &= T_1^3 - 3T_1T_2 + 2T_3\\
Q_4 &= T_1^4 - 6T_1^2T_2 + 3T_2^2 + 8T_1T_3 - 6T_4\\
Q_5 &= T_1^5 - 10T_1^3T_2 + 15T_1T_2^2 + 20T_1^2T_3 - 20T_2T_3 - 30T_1T_4 + 24T_5\\
Q_6 &= T_1^6 - 15T_1^4T_2 + 45T_1^2T_2^2 - 15T_2^3 + 40T_1^3T_3 - 120T_1T_2T_3\\
&\quad + 40T_3^2 - 90T_1^2T_4 + 90T_2T_4 + 144T_1T_5 - 120T_6\\
&\ \vdots
\end{aligned} \tag{184}$$

Worthy of special note is the *universality property* that attaches to the preceding formulæ: they read the same whatever the dimension N, but numerically

$$Q_{n>N} = 0 \text{ as a consequence of the Cayley-Hamilton theorem} \tag{185}$$

[56] I borrow my Q-notation from "A mathematical note: alogorithm for the efficient evaluation of the trace of the inverse of a matrix" (1996) so that I can most smoothly borrow certain results from that same source.

Of even greater relevance to my intended application is the fact that if $\mathbb{A}$ is "$\mathbb{M}$-antisymmetric"

$$A_{kj} = -A_{jk} \;\Leftrightarrow\; (\mathbb{M}\mathbb{A})^{\mathsf{T}} = -\mathbb{M}\mathbb{A} \;\Leftrightarrow\; \mathbb{W}\mathbb{A}^{\mathsf{T}}\mathbb{M} = -\mathbb{A}$$

then by an easy argument $\mathrm{tr}\mathbb{A}^n = (-)^n \cdot \mathrm{tr}\mathbb{A}^n$ which supplies

$$T_{\text{odd}} = 0 \tag{186}$$

and (184) simplifies very greatly:

$$\begin{aligned} Q_0 &= 1 \\ Q_1 &= 0 \\ Q_2 &= -T_2 \\ Q_3 &= 0 \\ Q_4 &= +3(T_2^2 - 2T_4) \\ Q_5 &= 0 \\ Q_6 &= -15(T_2^3 - 6T_2T_4 + 8T_6) \\ &\;\vdots \end{aligned} \tag{187}$$

EXAMPLE: Let us, in the case $N = 4$, (i) identify $\mathbb{M}$ with the Lorentz metric

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(ii) recognize that the g-orthogonal transformation matrices $\mathbb{U}$ have become Lorentz matrices (descriptive of Lorentz transformations, and often notated Λ) and (iii) identify $\mathbb{A}$ with the electromagnetic field tensor[57]

$$\mathbb{A} \quad \text{becomes} \quad \mathbb{F} \equiv \|F^\mu{}_\nu\| = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

Mathematica's

```
Tr[MatrixPower[F,2]]//Expand
```

and

```
Tr[MatrixPower[F,2]]^2-2Tr[MatrixPower[F,4]]//Expand
```

[57] See PRINCIPLES OF CLASSICAL ELECTRODYNAMICS (2001/2002), page 108.

commands instantly supply

$$Q_2 = -2\{\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B}\}$$
$$Q_4 = -24\{\boldsymbol{E}\cdot\boldsymbol{B}\}^2$$

which are familiar as *Lorentz invariant properties of the electromagnetic field tensor*.[58] It would appear on this evidence that we can expect invariants to be objects of physical importance.[59] From (182) we expect, in the case $N = 4$, to have $\frac{1}{4!}Q_4 = \det\mathbb{A}$, and indeed: we find by computation that[60]

$$\det\mathbb{F} = \{\boldsymbol{E}\cdot\boldsymbol{B}\}^2$$

Of course, it is clear on tensor-theoretic grounds that the traces of *all* powers of the field tensor (all contractions $F^\mu{}_{\alpha_1}F^{\alpha_1}{}_{\alpha_2}F^{\alpha_2}{}_{\alpha_3}\cdots F^{\alpha_p}{}_\mu$) are invariant. But

- $T_{\text{odd}} \equiv \text{tr}\,\mathbb{F}^{\text{odd}} = 0$
- $T_6, T_8, T_{10}, \ldots$ are redundant with T_2 and T_4 in consequence of the Cayley-Hamilton theorem, which in the electromagnetic instance reads

$$\mathbb{F}^4 - \{\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B}\}\,\mathbb{F}^2 - \{\boldsymbol{E}\cdot\boldsymbol{B}\}^2\,\mathbb{I} = \mathbb{O}$$

  and can be verified by (instantaneous) calculation.

I describe now an alternative appropach to expansion of $p(\lambda) \equiv \det(\mathbb{A} - \lambda\mathbb{I})$ which offers computational advantages in my intended application. Laplace would have us develop $\det\mathbb{A}$ by expansion along some arbitrarily selected row or column. An inherently more symmetrical procedure was devised by Cayley.[61] It involves "expansion along the principal diagonal," and is most simply explained by example:

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}
&= \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix}
+ a_{11}\begin{vmatrix} 0 & a_{23} & a_{24} \\ a_{32} & 0 & a_{34} \\ a_{42} & a_{43} & 0 \end{vmatrix} \\
&+ a_{22}\begin{vmatrix} 0 & a_{13} & a_{14} \\ a_{31} & 0 & a_{34} \\ a_{41} & a_{43} & 0 \end{vmatrix}
+ a_{33}\begin{vmatrix} 0 & a_{12} & a_{14} \\ a_{21} & 0 & a_{24} \\ a_{41} & a_{42} & 0 \end{vmatrix}
+ a_{44}\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} \\
&+ a_{11}a_{22}\begin{vmatrix} 0 & a_{34} \\ a_{43} & 0 \end{vmatrix}
+ a_{11}a_{33}\begin{vmatrix} 0 & a_{24} \\ a_{42} & 0 \end{vmatrix}
+ a_{11}a_{44}\begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix} \\
&+ a_{22}a_{33}\begin{vmatrix} 0 & a_{14} \\ a_{41} & 0 \end{vmatrix}
+ a_{22}a_{44}\begin{vmatrix} 0 & a_{14} \\ a_{41} & 0 \end{vmatrix}
+ a_{33}a_{44}\begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} \\
&+ a_{11}a_{22}a_{33}|0| + a_{11}a_{22}|0|a_{44} + a_{11}|0|a_{33}a_{44} + |0|a_{22}a_{33}a_{44} + a_{11}a_{22}a_{33}a_{44}
\end{aligned}$$

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[58] See page 184 in the notes just cited.

[59] In view of the importance of the role assigned by plane waves in electromagnetic theory it is interesting that *both invariants vanish* for those specialized solutions of Maxwell's equations.

[60] That $\det\mathbb{F}$ is a *perfect square* is no accident: it is a particular instance of a general circumstance to which we will soon attach major importance.

[61] See §125 in Thomas Muir, *A Treatise on the Theory of Determinants* (1928), which was reprinted by Dover in 1960.

The first determinant on the right has been "invertebrated." The second is the invertebrate (Sylvester's terminology) from which the 1^{st} row & column have been struck. At the sixth term on the right $a_{11}a_{22}$ multiplies the invertebrate from which the 1^{st} and 2^{nd}rows & columns have been struck. So it goes: on the right we find invertebrates multiplied by *diagonal elements taken in all possible combinations*.

Look now to $\det(\mathbb{A} - \lambda\mathbb{I})$ in a case in which $\mathbb{A}$ is antisymmetric. Writing $\mu \equiv -\lambda$ simply to avoid some distracting minus signs, we have

$$\begin{aligned}\begin{vmatrix} \mu & a_{12} & a_{13} & a_{14} \\ a_{21} & \mu & a_{23} & a_{24} \\ a_{31} & a_{32} & \mu & a_{34} \\ a_{41} & a_{42} & a_{43} & \mu \end{vmatrix} &= \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} + \mu\left\{ \begin{vmatrix} 0 & a_{23} & a_{24} \\ a_{32} & 0 & a_{34} \\ a_{42} & a_{43} & 0 \end{vmatrix} \right. \\ &\quad \left. + \begin{vmatrix} 0 & a_{13} & a_{14} \\ a_{31} & 0 & a_{34} \\ a_{41} & a_{43} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{14} \\ a_{21} & 0 & a_{24} \\ a_{41} & a_{42} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} \right\} \\ &\quad + \mu^2\left\{ \begin{vmatrix} 0 & a_{34} \\ a_{43} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{24} \\ a_{42} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix} \right. \\ &\quad \left. + \begin{vmatrix} 0 & a_{14} \\ a_{41} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{14} \\ a_{41} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} \right\} + \mu^4\end{aligned}$$

But easily, if $\mathbb{A}^{\mathsf{T}} = -\mathbb{A}$ then $\det\mathbb{A} = 0$ in all odd-dimensional cases, while (less obviously) in all even-dimensional cases $\det\mathbb{A}$ is a perfect square—the square of the so-called "Pfaffian" of $\mathbb{A}$:

$$\det\mathbb{A} = \begin{cases} 0 & \text{if antisymmetric } \mathbb{A} \text{ is odd-dimensional} \\ (\text{Pf}\,\mathbb{A})^2 & \text{if antisymmetric } \mathbb{A} \text{ is even-dimensional} \end{cases}$$

The terms linear in μ therefore drop away: we are left with

$$\begin{aligned}\det(\mathbb{A} - \lambda\mathbb{I}) = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 \\ - (a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2)\lambda^2 + \lambda^4\end{aligned}$$

We verify computationally that indeed

$$\begin{aligned}(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 &= \tfrac{1}{4!}Q_4 = \frac{3(T_2^2 - 2T_4)}{4!} \\ -(a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2) &= \tfrac{1}{2!}Q_2 = -\frac{T_2}{2!}\end{aligned}$$

The point is that those and similar expressions are <u>much easier to compute (by hand, if not in the opinion of *Mathematica*) by the Cayley-Pfaff method</u> than by assembling traces of powers.

So much for methodological preliminaries, for "the basic idea." Turning now to the specific questions posed on page 77, we have learned that we can find natural dwelling places for *multiple* invariants if we associate the pure elements of $\mathcal{C}_4[\boldsymbol{g}]$ with tensors off higher order than the vectors contemplated at (177). It seems most natural to consider "tensors of higher order" to mean "tensors of second rank," which are representable by square *matrices*. Look first to the numerology:

- The pure elements of $\mathcal{C}_4[\boldsymbol{g}]$ are $2^4 - 1 = 15$ component objects.
- Similarity transformations within $\mathcal{C}_4[\boldsymbol{g}]$ possess 3 invariants.

We are mindful that

- the space of <u>traceless 4×4 hermitian matrices</u> is $4^2 - 1 = 15$ dimensional and, additionally, that
- such objects do support a population of 3 real unitary invariants: they can be taken to be[62]

$$\begin{aligned} Q_2 &= -\ T_2 \\ Q_3 &= \ \ 2T_3 \\ Q_4 &= \ \ 3T_2^2 - 6T_4 \end{aligned}$$

But down that road lies a metric generalization of the standard **theory of Dirac matrices** which, valuable though it is, is not my destination.

We will instead proceed from the observation that

- the space of <u>real 6×6 antisymmetric matrices</u> is $1+2+3+4+5=15$ dimensional, and
- supports a population of 3 real rotational invariants, which at (187) were called Q_2, Q_4 and Q_6.

The immediate question: How to deploy the coordinates $\{V,T,A,P\}$ of $\boldsymbol{G}\in\mathcal{C}_4[\boldsymbol{g}]$ among the slots provided by such matrix? In the GENEVA NOTEBOOK I was guided by the fact that I was concerned there with the relationship between $\mathcal{C}_4[\boldsymbol{g}_{\text{Euclidean}}]$ and the "E-numbers"

$$\boldsymbol{E}_{\mu\nu} = -\boldsymbol{E}_{\nu\mu} \quad : \quad \mu,\nu\in\{0,1,2,3,4,5\}$$

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[62] Read from (184) with T_1 set equal to zero. It is instructive to write

$$\mathtt{H} = \begin{pmatrix} a_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} & a_{14}+ib_{14} \\ a_{12}-ib_{12} & a_{22} & a_{23}+ib_{23} & a_{24}+ib_{24} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_{33} & a_{34}+ib_{34} \\ a_{14}-ib_{14} & a_{24}-ib_{24} & a_{34}-ib_{34} & -(a_{11}+a_{22}+a_{33}) \end{pmatrix}$$

and then to ask *Mathematica* to execute the commands

```
CharacteristicPolynomial[H,λ]
ComplexExpand[%]
Simplify[%]
```

One obtains $\det(\mathtt{H}-\lambda\mathtt{I}) = \lambda^4 + 0\lambda^3 + \frac{1}{2!}Q_2\lambda^2 - \frac{1}{3!}Q_3\lambda^1 + \frac{1}{4!}Q_4\lambda^0$ and finds that all the Q's are indeed real: the i's have done no damage.

to which A. S. Eddington (1882–1944) assigned central importance in the work of his final years.[63] But even in the absence of any such "Eddingtonian bias" it seems entirely natural to construct

$$\begin{pmatrix} 0 & \square & \square & \square & \square & P \\ & 0 & T^{12} & T^{13} & T^{14} & \square \\ & & 0 & T^{23} & T^{24} & \square \\ & & & 0 & T^{34} & \square \\ & & & & 0 & \square \\ & & & & & 0 \end{pmatrix}$$

where the only remaining question is where to insert V's, where A's, and this is matter that can be settled by experimentation. Thus was I led to construct

$$\|G^{ab}\| \equiv \begin{pmatrix} 0 & A^1 & A^2 & A^3 & A^4 & P \\ & 0 & T^{12} & T^{13} & T^{14} & V^1 \\ & & 0 & T^{23} & T^{24} & V^2 \\ & & & 0 & T^{34} & V^3 \\ & & & & 0 & V^4 \\ & & & & & 0 \end{pmatrix} \tag{188.1}$$

and

$$\|\gamma_{ab}\| \equiv \begin{pmatrix} g & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} & g_{14} & 0 \\ 0 & g_{21} & g_{22} & g_{23} & g_{24} & 0 \\ 0 & g_{31} & g_{32} & g_{33} & g_{34} & 0 \\ 0 & g_{41} & g_{42} & g_{43} & g_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{188.2}$$

from which it follows that

$$\mathbb{G} \equiv \|G^a{}_b\| \equiv \|G^{ak}\gamma_{kb}\| = \begin{pmatrix} 0 & A_1 & A_2 & A_3 & A_4 & -P \\ -gA^1 & 0 & T^1{}_2 & T^1{}_3 & T^1{}_4 & -V^1 \\ -gA^2 & T^2{}_1 & 0 & T^2{}_3 & T^2{}_4 & -V^2 \\ -gA^3 & T^3{}_1 & T^3{}_2 & 0 & T^3{}_4 & -V^3 \\ -gA^4 & T^4{}_1 & T^4{}_2 & T^4{}_3 & 0 & -V^4 \\ -gP & -V_1 & -V_2 & -V_3 & -V_4 & 0 \end{pmatrix}$$

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[63] See *Relativity Theory of Protons & Electrons* (1936)—especially Chapter 2: "The sixteenfold frame"—and *Fundamental Theory*, which was published (1948) posthumously by E. T. Whittaker (who gave the book its unfortunate title). See also N. B. Slater, *The Development & Meaning of Eddington's 'Fundamental Theory'* (1957), which provides an account of the substance of Eddington's unpublished manuscripts, and of what light they may cast upon the evolution of his thought. Slater, by the way, was a friend of Eddington's and—during the period (1955/56) when he was working on his book—my frequent dining companion at Cornell, where I was a first-year graduate student and he a visiting scholar.

A few moments with pen and paper are sufficient to establish that (quoting now from (187), and writing $\mathcal{T}_n \equiv \text{tr}\,\mathbb{G}^n$ to emphasize that all traces are understood to refer specifically to the 6×6 matrix $\mathbb{G}$)

$$\begin{aligned} Q_2 &= -\mathcal{T}_2 \\ &= -\big\{V_m V^m + \tfrac{1}{2}T^m{}_n T^n{}_m - gA_n A^n + gP^2\big\} \\ &= \tfrac{1}{2}N_2 \quad : \quad \text{see again (170)} \end{aligned}$$

It was the prospect of such a result that guided me in the definitions (188). Discussion of the anticipated relationships that link

$$\begin{aligned} Q_4 &= \quad 3(\mathcal{T}_2^2 - 2\mathcal{T}_4) \\ Q_6 &= -15(\mathcal{T}_2^3 - 6\mathcal{T}_2\mathcal{T}_4 + 8\mathcal{T}_6) = 6!\det\mathbb{G} \end{aligned}$$

to the invariants N_3 and N_4 of (170) is computationally more burdensome. It is accomplished in the GENEVA NOTEBOOK by pen-and-paper work based upon the methods described on pages 78–81. *Mathematica* stands ready to assist, but by unfortunate quirk reads superscripts as powers. To work around this difficulty I proceed step-wise:

PHASE ONE: Assume the metric g_{ij} to be Euclidean, so that sub/superscript distinctions are irrelevant. Define

$$\texttt{T[m,n]:=}\ \frac{T_{m,n} - T_{n,m}}{2}$$

so as to inform *Mathematica* that T_{mn} is antisymmetric, enter

$$\texttt{G} = \begin{pmatrix} 0 & A_1 & A_2 & A_3 & A_4 & -P \\ -A_1 & 0 & T[1,2] & T[1,3] & T[1,4] & -V^1 \\ -A_2 & T[2,1] & 0 & T[2,3] & T[2,4] & -V^2 \\ -A_3 & T[3,1] & T[3,2] & 0 & T[3,4] & -V^3 \\ -A_4 & T[4,1] & T[4,2] & T[4,3] & 0 & -V^4 \\ -P & -V_1 & -V_2 & -V_3 & -V_4 & 0 \end{pmatrix}$$

command `Det[G]//Expand` and get a very long expression which, however, the command `Simplify[%]` brings to the form

$$\det\mathbb{G} = -\tfrac{1}{16}\big(\text{relatively brief 3}^{\text{rd}}\text{-order expression}\big)^2$$

Now command

$$\sum_{k=1}^4\sum_{l=1}^4\sum_{m=1}^4\sum_{n=1}^4 \texttt{Signature[\{k,l,m,n\}]} \Big(PT[k,l] + 2(A_kV_l - A_lV_k)\Big)T[m,n]\texttt{//Simplify}$$

and get -2(same "relatively brief 3^{rd}-order expression"), and conclude that

$$\tfrac{1}{6!}Q_6 = \det\mathbb{G} = -\tfrac{1}{16}\big(-\tfrac{1}{2}N_3\big)^2$$

whence

$$Q_6 = -\tfrac{45}{4}N_3^2$$

The invariant N_4 was seen at (170) to be quartic, and to contain a P^4 term. The Q_4-inspired command

$$3\Big(\texttt{Tr[MatrixPower[Gam, 2]]}^2\texttt{-2Tr[MatrixPower[Gam, 4]]}\Big)\texttt{//Expand}$$

gives, on the other hand, an expression that, though quartic, presents no P-powers higher than P^2. We are led therefore to construct

$$N_4 - \tfrac{1}{4}N_2^2 \;\;:\;\; \text{quartic, no } P^4 \text{ term according to (170)}$$

and by command

$$\texttt{Solve[}N_4 - \tfrac{1}{4}N_2^2 + xQ_4 \texttt{ == 0, } x\texttt{]//Simplify}$$

obtain $x = \frac{1}{6}$. Pulling these results together, we have (in the Euclidean case)

$$\left.\begin{aligned} Q_2 &= \quad\tfrac{1}{2}\,N_2 \\ Q_4 &= -\,6\,N_4 + \tfrac{3}{2}N_2^2 \\ Q_6 &= -\tfrac{45}{4}N_3^2 \end{aligned}\right\} \tag{189}$$

PHASE TWO is addressed to the demonstration that (189) hold even after the Euclidean assumption is abandoned. Let `G`, its subscripts notwithstanding, be understood to to mean $\|G^{ab}\|$. Enter

$$\texttt{g[m,n]:=}\;\frac{g_{m,n}+g_{n,m}}{2}$$

into the keyboard construction of $\|\gamma_{ab}\|$. Multiply those matrices as indicated on paged 83, to inform *Mathematica* what we have in mind when we write $\mathbb{G}$. Use $\mathbb{G}$ to construct the trace representations (187) of Q_2, Q_4 and $Q_6 = 6!\det\mathbb{G}$. Write

$$N_2 = -2\sum_{a=1}^{4} g[m,a]\big(V_aV_m - gA_aA_m\big) - \sum_{a=1}^{4}\sum_{b=1}^{4} g[m,a]g[n,b]T_{a,b}T_{m,n} - 2gP^2$$

to describe N_2, and provide similarly detailed descriptions of N_2 and N_4. All then proceeds as before... but slowly, even when *Mathematica 5* runs at several GHz, for the calculations are immense.

What we have established is that

- N_2, N_3 and N_4 are invariant under similarity transformations of the sort (see again 174) and (178.1) encountered within $\mathcal{C}_4(\mathscr{g})$:

$$\boldsymbol{G} \longmapsto \boldsymbol{G}' = e^{-\theta\boldsymbol{H}}\boldsymbol{G}e^{\theta\boldsymbol{H}}$$

- a distinct but equivalent set of objects Q_2, Q_4 and Q_6 arises when one looks to the response of 6-dimensional antisymmetric tensors G^{pq}

to γ-orthogonal transformations. The argument—a refinement of that encountered already on page 77—runs as follows: let

$$G^{pq} \longmapsto G'^{pq} = U^p{}_a G^{ab} U^q{}_b$$

be notated

$$\varGamma \longmapsto \varGamma' = \mathbb{U}\varGamma\,\mathbb{U}^{\mathsf{T}}$$

and assume $\mathbb{U}$ to be γ-orthogonal: $\gamma = \mathbb{U}^{\mathsf{T}}\gamma\,\mathbb{U}$. Then $\varGamma' = \mathbb{U}\varGamma\gamma\mathbb{U}^{-1}\gamma^{-1}$ becomes

$$\mathbb{G}' = \mathbb{U}\mathbb{G}\mathbb{U}^{-1}$$

with $\mathbb{G} \equiv \varGamma\gamma = \|G^{pa}\gamma_{aq}\| \equiv \|G^p{}_q\|$. Clearly $\mathbb{G}'$ and $\mathbb{G}$ share the same characteristic polynomial, which by the assumed antisymmetry of $\varGamma$ (γ-antisymmetry of $\mathbb{G}$) has the form

$$\det(\mathbb{G} - \lambda\mathbb{I}) = \lambda^6 + \tfrac{1}{2!}Q_2\lambda^4 + \tfrac{1}{4!}Q_4\lambda^2 + \tfrac{1}{6!}Q_6\lambda^0$$

Which brings us to the core of the matter: if $\mathbb{U} = e^{-\beta\mathbb{H}}$ is γ-orthogonal then (as argued already on page 77) $\mathbb{H}$ is necessarily γ-antisymmetric. Assuming β to be infinitesimal, we have

$$\mathbb{G}' = \mathbb{G} + \beta[\mathbb{G}\mathbb{H} - \mathbb{H}\mathbb{G}] + \cdots$$

The γ-antisymmetry of $\mathbb{G}$ and $\mathbb{H}$ implies that of $[\mathbb{G}\mathbb{H} - \mathbb{H}\mathbb{G}]$. So—taking $\mathbb{G}$ to have the design indicated on page 83, and $\mathbb{H}$ to be[64] the lower case version of that matrix—to we can write

$$\mathbb{G}\mathbb{H} - \mathbb{H}\mathbb{G} = \begin{pmatrix} 0 & \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 & -\mathcal{P} \\ -g\mathcal{A}^1 & 0 & \mathcal{T}^1{}_2 & \mathcal{T}^1{}_3 & \mathcal{T}^1{}_4 & -\mathcal{V}^1 \\ -g\mathcal{A}^2 & \mathcal{T}^2{}_1 & 0 & \mathcal{T}^2{}_3 & \mathcal{T}^2{}_4 & -\mathcal{V}^2 \\ -g\mathcal{A}^3 & \mathcal{T}^3{}_1 & \mathcal{T}^3{}_2 & 0 & \mathcal{T}^3{}_4 & -\mathcal{V}^3 \\ -g\mathcal{A}^4 & \mathcal{T}^4{}_1 & \mathcal{T}^4{}_2 & \mathcal{T}^4{}_3 & 0 & -\mathcal{V}^4 \\ -g\mathcal{P} & -\mathcal{V}_1 & -\mathcal{V}_2 & -\mathcal{V}_3 & -\mathcal{V}_4 & 0 \end{pmatrix} \tag{190}$$

where by straightforward calculation

$$\left.\begin{aligned} \mathcal{V}^i &= \Big\{ -t^i{}_m V^m - v_m T^{mi} - gpA^i + ga^i P \Big\} \\ \mathcal{T}^{ij} &= \tfrac{1}{2}\Big\{ -(v^iV^j - v^jV^i) - (t^i{}_mT^{mj} - t^j{}_mT^{mi}) + g(a^iA^j - a^jA^i) \Big\} \\ \mathcal{A}^j &= \Big\{ -pV^j - a_mT^{mj} - t^j{}_mA^m + v^jP \Big\} \\ \mathcal{P} &= \Big\{ -a_mV^m + v_mA^m \Big\} \end{aligned}\right\} \tag{191}$$

Comparison with (176) on page 76 establishes that if $\boldsymbol{G} \leftrightarrow \mathbb{G}$ (in the sense "share the same $\{V, T, A, P\}$ coefficients) and if also $\boldsymbol{H} \leftrightarrow \mathbb{H}$, then

$$\boldsymbol{GH} - \boldsymbol{HG} \;\longleftrightarrow\; 2(\mathbb{G}\mathbb{H} - \mathbb{H}\mathbb{G}) \tag{192}$$

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[64] See again (151) on page 63.

. . . from which follows the important conclusion that

$$\boldsymbol{G} \longmapsto \boldsymbol{G}' = e^{-\frac{1}{2}\theta \boldsymbol{H}}\boldsymbol{G}\, e^{\frac{1}{2}\theta\boldsymbol{H}} \tag{193.1}$$

and

$$\mathbb{G} \longmapsto \mathbb{G}' = e^{-\theta\,\mathbb{H}}\mathbb{G}\, e^{\theta\,\mathbb{H}} \tag{193.2}$$

<u>achieve the same action</u>: $\{V, T, A, P\} \longmapsto \{V', T', A', P'\}$. And (193.2)—because $e^{-\theta\,\mathbb{H}}$ is γ-orthogonal—can be phrased

$$G^{pq} \longmapsto G'^{pq} = \big(e^{-\theta\,\mathbb{H}}\big)^p{}_a\big(e^{-\theta\,\mathbb{H}}\big)^q{}_b\, G^{ab} \tag{193.3}$$

which is to say: similarity transformations within $\mathcal{C}_4(\boldsymbol{g})$ are equivalent to the γ-orthogonal transformations of antisymmetric *tensors* in 6-space. It becomes natural in view of (193.3) to look to the transformation of contravariant 6-*vectors*

$$\xi^p \longmapsto \xi'^p = \big(e^{-\theta\,\mathbb{H}}\big)^p{}_a\,\xi^a \tag{194}$$

where γ-orthogonality entails

$$\xi'^p\gamma_{pq}\,\xi'^q = \xi^p\gamma_{pq}\,\xi^q \tag{195}$$

It will be appreciated that we have in (193.2) a 6×6 matrix representation not of $\mathcal{C}_4(\boldsymbol{g})$ itself, but only of the associated *commutator sub-algebra*:

$$\boldsymbol{GH} = \tfrac{1}{2}(\boldsymbol{GH}+\boldsymbol{HG}) + \underbrace{\tfrac{1}{2}(\boldsymbol{GH}-\boldsymbol{HG})}_{\text{admits of 6-dimensional representation}}$$

When we to write

$$\mathbb{G} = S\,\mathbb{I} + V^i\,\mathbb{E}_i + \tfrac{1}{2}T^{ij}\,\mathbb{E}_{ij} + A^j\,\mathbb{F}_j + P\,\mathbb{F}$$

we find, for example, that

$$\mathbb{E}_i\mathbb{E}_j + \mathbb{E}_j\mathbb{E}_i \neq 2g_{ij}\,\mathbb{I}$$

9. Numerological source of interest in the twelfth-order theory. To identify a pure element of $\mathcal{C}_2$—a structure that can be considered to be rooted in the 2-dimensional group $O(2)$—one must assign value to $2^2 - 1 = 3$ coefficients. The physically important relationship between

- similarity transformations within $\mathcal{C}_2$ on the one hand, and
- the group $O(3)$ of rotations in 3-space on the other

owes something to the fact that $1 + 2 = 3$ is the number also of the matrix elements that must be specified to identify an antisymmetric 3×3 matrix (logarithm of a 3×3 rotation matrix). The 2-dimensionality of the irreducible complex matrix representations of $\mathcal{C}_2$ brings complex 2-vectors (simple spinors) into play as natural companions of real 3-vectors.[65] Similarly . . .

[65] And 2-spinors of higher rank into play as companions of real 3-tensors of higher rank. While we stress here the generative relation of $O(2)$ to $\mathcal{C}_2$, it should be borne in mind that $O(2)$ has a "downwardly natural" relationship also to the complex numbers (rotations on the complex plane).

To identify a pure element of $\mathcal{C}_4$—a structure that can be considered to be rooted in the 2-dimensional group $O(4)$—one must assign value to $2^4 - 1 = 15$ coefficients. The relationship—developed above—between

- similarity transformations within $\mathcal{C}_4$ on the one hand, and
- the group $O(6)$ of rotations in 6-space on the other

owes something to the fact that $1+2+3+4+5 = 15$ is the number also of the matrix elements that must be specified to identify an antisymmetric 6×6 matrix (logarithm of a 6×6 rotation matrix). The 4-dimensionality of the irreducible complex matrix representations of $\mathcal{C}_4$ brings complex 4-spinors into play as natural companions of real 4-vectors/tensors. Physical importance attaches familiarly to the scalar/vector/tensor/pseudovector/pseudoscalar latent in the transform theory of $\mathcal{C}_4$ (Dirac algebra). Less familiar is the demonstrated fact that those objects are latent also in the theory of $O(6)$, and no work (so far as I am aware) has been assigned by physicists to the associated 6-vectors,[66] or to 6-tensors of higher order.

It was with these points in mind that, in 1967, I was led to ask: Are there yet other instances in which a Mersenne number $2^p - 1$ is triangular? Are there higher instances of

$$2^p - 1 = \sum_{k=1}^{n-1} k = \tfrac{1}{2}n(n-1) = \tbinom{n}{2} \tag{196.1}$$

We have already in hand the cases

$$\begin{aligned} 2^1 - 1 &= \sum_{k=1}^{2-1} k = 1 \\ 2^2 - 1 &= \sum_{k=1}^{3-1} k = 3 \\ 2^4 - 1 &= \sum_{k=1}^{6-1} k = 15 \end{aligned}$$

Laborious work with a Frieden calculator exposed also the case

$$2^{12} - 1 = \sum_{k=1}^{91-1} k = 4095$$

Though further searching provided no additional examples, I recorded at the time my guess that "the number of triangular Mersennes is probably infinite." I consulted my then-colleagues in the Reed College Mathematics Department, and was informed by Burrows Hunt that "there are very few theorems refer to the intersection of sparse sequences." So there I left it...

[66] These are not to be confused with the 6-vectors that ion electrodynamics are sometimes associated with the antisymmetric 4×4 field tensor.

. . . until June 1972, when it came to my attention that Brian Tuckerman, of IBM, had (in 1971) discovered the 24th Mersenne prime.[67] I wrote to him to discover what he might tell me about triangular Mersenne numbers. Six days after my letter was posted he wrote back to remark (*i*) that the substitution $n = \frac{1}{2}(m+1)$ casts $2^p - 1 = \frac{1}{2}n(n-1)$ into the form

$$2^{p+3} - 7 = m^2 = (2n-1)^2 \tag{196.2}$$

(*ii*) that—except for the case $p = 1$—the p in (196) must certainly be even, and (*iii*) that he had searched up to $p = 10^5$ and found no solution beyond my $p = 12$. Tuckerman guessed that my problem must have been studied, and referred me to D. H. Lehmer (celebrated number theorist at Berkeley) for references. I wrote immediately to Lehmer, who (again within six days) reported that the list

$$(p, n) = \big\{(0,0), (1,2), (2,3), (4,6), (12,91)\big\}$$

is exhaustive!, as had been shown by D. J. Lewis in 1961, and that proof can be found also on pages 205–6 in L. J. Mordell's *Diophantine Equations* (1969).

Upon consulting Lewis[68] I learned that my $2^{p+3} - 7 = m^2$ problem is the simplest instance of a class of problems that can be shown "by means of a p-adic argument" to possess finitely many solutions. Reference is made to earlier papers by T. Nagell (1923, 1948 and 1954) and by Th. Skolem, S. Chowla & D.J. Lewis.[69] The latter begins with these words: "Ramanujan[70] observed that the equation $2^{n+2} - 7 = x^2$ has . . . integral solutions for $n = 1, 2, 3, 5, 13$; and he conjectured that these are the only solutions. The authors cite earlier work, but claim to be the first to establish the validity of Ramanujan's conjecture. That claim inspired an indignant T. Nagell to publish "The Diophantine equation $x^2 + 7 = 2^n$," Arkiv für Matematik **4**, 185 (year not recorded), in which he draws attention to the fact that proof of Ramanujan's conjecture appears as Problem 165 on page 272 of Nagell's *Introduction to Number Theory* (1951): Nagell then presents an English translation of his own "quite elementary" proof of 1948.

Upon consulting *Collected Papers*[70] we find that pages 322–334 record QUESTIONS & SOLUTIONS submitted by Ramanujan to the Journal of the Indian

[67] It is $2^{19937} - 1$ and runs to 6002 decimal digits. At present the largest known Mersenne prime—discovered only a few days ago (November 2003) and thought to be the 40th—is $2^{20996011} - 1$, which runs to 6,320,430 digits. The distributed calculations that have identified the last few Mersenne primes have made critical use of an algorithm devised by Richard Crandall.

[68] "Two classes of Diophantine equations," Pacific Journal of Mathematics **11**, 1063 (1961).

[69] "The Diophantine equation $2^{n+2} - 7 = x^2$ and related problems," Proc. Amer. Math. Soc. **10**, 663 (1959).

[70] G. H. Hardy *et al* (editors), *Collected Papers of Srinivasa Ramanujan* (1927), page 327, Problem 464.

Mathematical Society. QUESTION 464 reads "$2^n - 7$ is a perfect square for the values 3, 4, 5, 7, 15 of n. Find other values." This I don't read as a conjecture that there are *no* other values... but perhaps it can be argued that if there were other values Ramanujan would have had no interest in the problem.

So we have basically **three and only three cases**

$$\mathcal{C}_2 \longleftrightarrow O(3)$$
$$\mathcal{C}_4 \longleftrightarrow O(6)$$
$$\mathcal{C}_{12} \longleftrightarrow O(91)$$

in which the numerology works out. The first two can, in fact, be developed in detail, and are of established physical importance. The question therefore arises: *Can the "last case" be developed in similar detail, and has it a role to play in the description of the real world?*

Possibly relevant is the observation that 91 is itself triangular:

$$\begin{aligned} 1+2+\cdots+13 &= 91 \\ &= \text{number of elements in a } 14\times 14 \text{ antisymmetric matrix} \end{aligned}$$

This suggests that we might adopt antisymmetrized double indexing to describe the elements $x_{ij} = -x_{ji}$ $(i, j = 1, 2, \ldots, 14)$ of a 91-vector, and in that same (Eddingtonian) spirit write

$$A_{ij,kl} = \begin{cases} -A_{kl,ij} \\ -A_{ji,kl} \\ -A_{ij,lk} \end{cases}$$

to describe the 4095 elements of a 91×91 antisymmetric matrix. But how to make that convention mesh with the convention

$$\boldsymbol{e}_p\boldsymbol{e}_q + \boldsymbol{e}_q\boldsymbol{e}_p = 2g_{pq}\mathbf{I} \quad : \quad p, q = 1, 2, \ldots, 12$$

natural to the development of $\mathcal{C}_{12}$?